The geometrical machines, like pantographs for geometrical transformations, have revealed educational potentialities as tools for teaching and learning mathematical proof: they provide situations that can promote the formulation of conjectures and the generation of argumentations. In this paper, we propose a first analysis of the argumentations produced in activities with the machines. In particular, we show that the argumentation can refer to different elements of the pantograph (the structure, the movement, and the drawings traced by the machine) and to different components (figural and conceptual) of the geometrical figures representing the linkages.

INTRODUCTION

The Laboratory of Mathematical Machines (MMLab), at the Department of Mathematics of University of Modena and Reggio Emilia (Italy), contains a collection of instruments, called Mathematical Machines, which have been reconstructed with a didactical aim, according to the design described in historical texts. In this paper, we refer to the geometrical machine as a tool that forces a point to follow a trajectory or to be transformed according to a given law (Bartolini Bussi & Maschietto, 2008).

The MMLab organizes activities with the Mathematical Machines for secondary school students, groups of university students, pre-service and practicing school teachers (Maschietto & Martignone, 2008; Bartolini Bussi & Maschietto, 2008). The design and the development of these activities are carried out by the MMLab research group with the aim to provide a suitable learning context in which to activate important processes, such as the construction of meanings and the construction of proof (Bartolini Bussi, 2000).

Our research focuses on machines that establish a correspondence between points of the plan regions, like reflection, central symmetry, translation, rotation, and homothety. These transformations are physically performed through two leads fixed in two plotter points of an articulated system composed by some rigid rods and some pivots (see fig. 1). These machines, named pantographs or linkages, incorporate
some mathematical properties in such a way as to allow the implementation one geometrical transformation.

Fig.1: The Scheiner’s pantograph for the homothety

The study presented in this paper is part of a wider research about the didactical potentiality of the machines as tools for teaching and learning mathematical proof. Our goal is to investigate the cognitive processes involved in the proof construction in activities with the geometrical machines (some first results are in Martignone & Antonini, 2009a, 2009b): in order to do that, we study the processes that can lead to the construction of a proof or that can be an obstacle to this construction. In particular, we deal with the analysis of argumentations proposed by subjects to support their conjectures.

THEORETICAL FRAMEWORK

The study presented here focus on argumentation in geometry context. In the following sections we will expose some theoretical considerations on argumentation and proof and on the geometrical figures.

Argumentation and proof

Many papers have been written on the relationships between argumentation and proof (Mariotti 2006). In general, a mathematical proof of a statement consists of a logical sequence of propositions that states the validity of the statement. Differently, an argumentation consists of a rhetoric means that have the goal to convince somebody of the truth or the falsehood of a statement.

Some authors focus on the differences between argumentation and proof (see, for example, Duval, 1992-93). On the other hand, focussing on the processes of argumentation and proof generation, the theoretical framework of Cognitive Unity (Garuti et al., 1996; Mariotti et al., 1997; Garuti et al., 1998; Pedemonte, 2002; Boero, 2007), without forgetting the differences, underlines the analogies between
them. In particular, these studies suggest that, in open-ended problems (where students are asked to produce a conjecture, to generate an argumentation and a proof that support the conjecture), a continuity between the argumentation and the subsequent mathematical proof may or may not occur. For these reasons, it is important to identify the factors that can favour continuities and the elements that can lead to a gap between argumentation and proof.

One goal of our research project is to investigate the continuity between argumentation and proof during the machine exploration that led to the identification of the mathematical law made by the machine. In this paper, we focus on the argumentations, in the particular situation in which students are involved with pantographs or linkages for geometrical transformations of the plane.

We underline that a mathematical proof is necessarily linked to a theory, a theoretical framework within which the proof makes sense (see Mariotti et al., 1997), while an argumentation in geometry can also concern some figural aspects (as magnitude, shape, etc.) without reference to a theory. In order to take into account these aspects we refer to the theoretical framework of figural concepts.

The theory of figural concepts

The theory of figural concepts (Fischbein, 1993) provides us with an efficient theoretical tool, suitable to analyse cognitive processes in geometrical problem solving. According to Fischbein (1993), mental entities involved in geometrical reasoning cannot be considered either pure concept or mere image. Geometrical figures are mental entities that simultaneously possess both conceptual properties (as general propositions deduced in the Euclidean theory) and figural properties (as shape, position, magnitude). Fischbein called them figural concepts. A productive reasoning, as an efficient process of proof generation, can be generally explained by the fact that the figural and the conceptual aspect blend in a figural concepts (see, for example Mariotti, 1993; Mariotti & Fischbein, 1997). Our analysis of the argumentation will take into account the distinction and the duality between the components of a figural concept.

METHODOLOGY

The goal of our research is to study the argumentation generated by subjects that are asked to discover what the machine does and to prove it. In this first step of the research we needed to analyse rich argumentation activities linked to the particular situation in which a mathematical machine are the object that has to be explored.

For this reason, we selected subjects who were familiar with geometry and with problem-solving but that have not seen these machines before. The subjects were three pre-service teachers, two university students and one young researcher in mathematics. The task was identifying the geometrical transformation and to prove that the machine performed that transformation. Data were collected through clinical
interviews that were videotaped. Subjects were asked to express their thinking process aloud. The analysis of the interviews is based both on the transcripts and on the manipulative activities on the pantographs. In this paper we report some results about the explorations of a linkage for reflection (fig. 2) and of a pantograph for homothety (Scheiner’s Pantograph, see fig. 3).

Fig. 2: An image of pantograph for reflection and its products. Two opposite vertices (A and B) of a rhombus, composed of four equal rods pivoted together, can move in a groove (a straight path $r$). The other opposite vertices (P and Q) are corresponding in the reflection of axis $r$.

Fig.3: An image from Scheiner’s pantograph. Four rods are pivoted so that they form a parallelogram AQCB. The point O is pivoted on the plane. It is possible to prove that P, Q and O are on the same line and that P and Q correspond in the homothety of centre O and ratio BO/OA.
ARGUMENTATION IN EXPLORING THE GEOMETRICAL MACHINES

The argumentations justifying that the machine implements a particular geometrical transformation are closely related to three elements: the drawings traced by the machine, the structure of the machine (as a figural or a conceptual component), and the machine movement. Notice that the same subject could propose more than one type of argumentation. This is common in tasks requiring the production of a conjecture and a proof, when one arguments with different goals: producing, testing, supporting, and proving a conjecture.

Argumentation that refers to the drawings traced by the machine

These argumentations refer to the shape of the drawings traced by the machine and to their comparison. There are no theoretical references in this type of argumentation. The machine is used to perform effectively the transformation and the argumentation is based on the products of the transformation. We notice that the drawings can be traced by the leads of the machine but there is also the possibility that the subject sees the drawings only through the movement of the plotter points.

For example, some of the pre-service teachers, exploring the Scheiner’s pantograph, state that the machine performs a homothety because one of the two drawings traced by the machine appears as an enlargement of the other one.

Argumentation that refers to the structure of the machine

These argumentations refer to one or more elements of the static structure of the machine, that is the structure of the machine when it is stopped in some position. We give two examples. In the first, the subject refers to the figural aspect of the structure, in the second to the conceptual aspect of the geometrical figure represented by the articulated system.

Example 1. Lucia, a pre-service teacher, exploring the Scheiner’s pantograph, conjectures that the transformation is a homothety and she justifies her statement saying that “it has these two pivots [she points at A and B], and this rod [BP] is longer than this [AQ]”. This, for Lucia, explains the fact that the drawing made by the lead put in the point Q is an enlargement of the drawing traced by the lead put in the point P. According to the theory of figural concepts, the argumentation refers to the figural aspect (a qualitative relationship between the length of BP and the length of AQ) of the geometric figure represented by the articulated system, without any reference to a mathematical theory. In a following section we will present a deeper analysis of another example.

Example 2. Anna, a pre-service teacher, exploring the linkage for reflection (fig. 2), justifies that the transformation is a reflection showing that the two plotter points (P and Q) are on the segment whose groove is the perpendicular bisector. Then Anna refers to some conceptual properties of rhombus to support this fact. According to the theory of figural concepts, the argumentation refers to the conceptual aspects of
the geometric figure represented by the articulated system. Another example will shown in a following section.

**Argumentation that refers to the movement of the machine**

These argumentations refer to some dynamic properties of the articulated system, i.e. to some characteristics of its movement.

*Example.* In the transcript we will analyse below, Carlo proposes many argumentations to support his conjecture about the linkage for the reflection (fig. 2). One of these argumentations refers to the movement of the machine, in particular to the fact that if one plotter point approaches the reflection axis, then also the other plotter point approaches that (see below).

**ANALYSIS OF A TRANSCRIPT**

In this section, we analyse a transcript of an interview in which the subject proposes more than one argumentation to support his conjecture about the transformation made by the machine: one argumentation referring to the *figural aspect* of the structure, one referring to the *movement* and one referring to the *conceptual aspect* of the articulated system. Only after these argumentations he generates a mathematical proof. Carlo is a young researcher in mathematics and this is his first experience with mathematical machine. He is exploring the pantograph for the reflection.

Carlo: […] I see a line in the centre and I think to a symmetry (*he makes a gesture opening his hands in a symmetric way, like he is opening a book*). I have thought to a symmetry… reflection, because there is this line, it could not be, but… it’s all quite… so symmetric that… (*he makes a rhombus with his hands*)

Carlo considers the *figural aspects* of the machine structure. It is the shape, the symmetry, underlined by the groove in the centre, that starts to convince Carlo that the transformation is the reflection (*argumentation referring to the figural aspect of the structure of the machine*).

Carlo: […] well, so, then I remind that this will be the transformation […] symmetry… as it is called… reflection… I had seen it immediately for the shape.

Carlo recalls the previous argumentation and the fact that it refers to the *figural aspect* of the structure of the articulated system.

Carlo: Because these are rigid (*he points at the rods and then he starts to move the articulated system*) then probably there can be some properties linked to the… (*he stops the movement of the machine*)… rhombus.

The observation of the structure of the articulated system ("these are rigid") leads Carlo to anticipate that there could be some mathematical properties related to the figure now identified as rhombus. These properties can justify that the machine
implements the reflection. There is here an important transition, anticipated and not yet implemented, from an argumentation linked to the figural aspects of the machine structure to an argumentation related to the conceptual aspect of a geometrical figure (rhombus) that represents the articulated system. For the moment it is only an anticipation that will lead later to the construction of a mathematical proof.

Carlo: Anyway, now we will see…

Carlo postpones the generation of the proof: what he has said about the properties of the rhombus remains only an anticipation. As we can see below, it seems that he feels the need of other argumentations before constructing the proof.

Carlo: And also the movement (he starts again to move the machine with two hands), the movement seems to me quite significant for this: if I approach the axis with this point [P], also the other point [Q] approaches it, both perpendicularly … the fact that there is a movement also in this direction (he moves the articulated system by sliding the pins, the points A and B in the fig. 2, in the groove) …

This argumentation is linked to a property of the machine movement, showing a dynamic relationship between the two corresponding points (P and Q) of the transformation; for Carlo it supports again the fact that this is a reflection (argumentation referring to the movement of the machine).

Carlo: I do not need the leads.

He does not feel the need to use the leads, he does not consider important to have an argument based on the tracks. The different argumentations produced have convinced Carlo but probably the use of the leads is not taken into consideration also because he is aware that they would not give any new contribution to the knowledge of the machine functioning and then to the construction of a proof.

Carlo: Then, why it works … so if they all have the same length, we have a figure… this is a geometric figure with four equal sides … where this (he follows the groove with his finger) is a diagonal, therefore it's a rhombus … for the rhombus properties the two diagonals are perpendicular, this tells me that this diagonal (he tracks with his finger an imaginary diagonal that joints the pointer point with the tracer point) is perpendicular and also they (the two diagonals) bisect each other.

Here, Carlo proposes an argumentation based on the geometrical properties of the articulated system (argumentation referring to the conceptual aspect of the structure of the machine).

We observe that Carlo has no difficulties in the identification of the transformation. Nevertheless he needs to look for additional argumentations before constructing the proof. After a first argumentation based on the figural components of the structure of the machines, he proposes an argumentation referring to the relationships between
the movement of the two points involved in the transformation. Finally he comes back to the structure of the articulated systems but this time he considers the conceptual components, supporting his statement in the geometrical theory.

We think that the different argumentations proposed by Carlo have the goal of strengthening the conviction that the conjecture is true, but also to offer a more complete argumentation that can take into account different perspectives about the machine and its use. It is interesting to notice that, differently from students’ processes, the only type of argumentation not generated by Carlo is that linked to the drawings traced by the machine. This type of argumentation is very useful to generate a conjecture and to test it, and for this reason it is very common among students with minor experience. Nevertheless, as Carlo feels, it could not give contributions neither to the explanation of the movement nor to the proof construction.

CONCLUSIONS

The subjects, as we have shown in the Carlo’s protocol, produced different types of argumentations, even during the same exploration phase. It is important to underline that mathematical culture, but also familiarity with the machines, seem to promote the emergence and the development of different types of argumentations. In fact, we have noticed that the experts produce several argumentations and they refer to the drawings traced by the machine only when they have difficulties in identifying the mathematical law incorporated in the machine; differently, the argumentations referring to drawings are very common in students’ activities.

The proposed analysis of the machines explorations paves the way for the generation of hypotheses on the transition from argumentation to proof. In particular, we hypothesize that, in the case of argumentations referring to the conceptual part of the machine structure, there can be cognitive unity between argumentation and proof, in other cases it seems plausible to attend to cases of cognitive break. A special case seems to be the argumentations based on movement, because they often lead to further argumentations that explain the motion through the structure of the articulated system, and a cognitive unity may or may not occur.

Regarding the transition between argumentation and proof, we mention briefly that in the interviews we have carried on, the interviewer’s interventions has been relevant in guiding the students to the proof construction. In fact, it seems that some interventions, aimed to put an emphasis also on argumentations that students do not spontaneously generate, can be educationally effective in the activities with machines oriented to stimulate and to develop argumentative and proving processes.
REFERENCES


