THE TEACHING OF LINEAR EQUATIONS: COMPARING EFFECTIVE TEACHERS FROM THREE HIGH ACHIEVING EUROPEAN COUNTRIES

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On various international tests of achievement Finnish, Flemish and Hungarian students have been amongst the more successful in Europe. Linear equations, a topic students traditionally find difficult, is a key topic in the transition from mathematics as inductive and concrete to deductive and abstract. This paper, by means of an analysis of video-taped lessons taught by case study teachers, one from each of Finland, Flanders and Hungary, examines comparatively how teachers defined locally as effective construct opportunities for their students to learn the mathematics of linear equations. The findings show that all three teachers acted in ways contrary to received research wisdom, exploiting the balance scale as the key metaphor for inducting students into the solution processes of algebraic equations.

INTRODUCTION

The transition from arithmetic to algebra is problematic due, not least, to ambiguities with regard to the role and meaning of symbols of mathematics in general and the equals sign in particular. On the one hand it is a command to execute an operation, reflecting procedural (Kieran, 1992) or operational (Sfard, 1995) expectations. On the other, it is as an object on which other operations may be performed, reflecting structural (Sfard, 1995) expectations. Related to such concerns is the distinction between arithmetical equations and algebraic equations. The former, with the unknown on one side only, are generally assumed to be susceptible to undoing (Filloy & Rojano, 1989). However, the latter, with unknowns on both sides, cannot be solved by arithmetic-based approaches and require not only that the learners “understand that the expressions on both sides of the equals sign are of the same nature (or structure)” (Filloy & Rojano, 1989, p. 19) but also that they are able to operate on the unknown as an entity and not a number. In this manner, arithmetic equations are procedural while algebraic equations are structural (Kieran, 1992; Boulton-Lewis, Cooper, Atweh, Pillay, Wilson & Mutch, 1997). However, many students fail to navigate this transition and are “reduced to performing meaningless operations on symbols they do not understand” (Herscovics & Linchevki, 1994, p. 60. This failure has been described as either a cognitive gap (Herscovics & Linchevski 1994) or a didactic cut (Filloy & Rojano, 1989), although Pirie and Martin (1997) argue it is more likely to be the responsibility of inappropriate didactics than cognitive inadequacies.

Research shows that the use of different embodiments or representations can create the potential for new concepts, entities and operations to become endowed with meaning (Filloy & Rojano, 1989). They can facilitate the link between concrete and
abstract thinking by acting as analogues for the intended abstractions (English & Sharry, 1996; Warren & Cooper, 2005). With regard to equation solving, one of the most frequently used, and criticised, embodiments is the balance scale. Its advocates argue that it both helps students understand equations as entities rather than computational instructions and supports those symbolic representations that underpin algebraic formalisms (Filloy & Rojano, 1989; Warren & Cooper, 2005). Its critics argue that it cannot represent negatives in anything but a contrived way and is unfamiliar to modern students (Pirie & Martin, 1997).

Despite such criticisms several studies have examined the efficacy of the balance. Warren and Cooper (2005) found it helped children not only solve algebraic equations but also understand the equals sign as representing equivalence between entities. Vlassis (2002) found that although students understood both the conceptual and procedural role of the balance many experienced difficulties with negatives, irrespective of whether an equation was arithmetical or algebraic. Boulton-Lewis et al. (1997) found students preferred to use inverse approaches rather than the concrete representations taught them. Such examples highlight the diversity of findings with respect to the use of the balance, although their respective research designs may have contributed significantly to the findings. Warren and Cooper (2005) and Vlassis (2002) invoked the balance as a means of solving algebraic equations, while Boulton-Lewis et al. (1997) did so with arithmetic equations only.

In the following, the teaching of linear equations by three teachers, one from each of Finland, Flanders and Hungary is examined. Analyses of such teaching, located in countries typically shown to be more successful than England on various TIMSS and PISA assessments, present opportunities for an evaluation of the adaptive potential of the culturally located practices of one culture for another (Clarke, 2004). That is, such analyses have the potential to inform curriculum and teacher education development in less successful but culturally similar countries like England.

**METHOD**

This paper draws on data from the EU-funded, Mathematics Education Traditions of Europe (METE) project. Based at Cambridge, England, the project, which ran from 2003-2005, examined aspects of mathematics teaching in Flanders\(^1\), England, Finland, Hungary and Spain. This paper draws on analyses of video recordings of sequences of lessons taught by teachers defined locally as competent in the manner of the Learner’s Perspective Study (Clarke, 2006). Thus, while it is not possible to comment on the attainment of project students, it would be reasonable to assume that project teachers would be as successful as any of their national peers. Four sequences

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\(^1\) From the perspective of the METE study, Flanders, the autonomous Dutch-speaking community of Belgium, is construed as a country. This distinction is well known in the literature with, for example, Flanders being reported as a different educational system from Wallonia, the French-speaking community of Belgium.
of five lessons were filmed on the same topic in each country during the academic year 2003-2004. One of the topics was linear equations, chosen because of its importance in the transition from arithmetic to algebra. Videographers focused on the teacher whenever they were speaking. Teachers wore radio-microphones while a static microphone was placed strategically to capture as much student talk as possible. For each sequence, according to agreed project procedures, the first two lessons were transcribed and translated by English speaking colleagues working in the project universities. The translations, which were verified by Finnish-, Hungarian- and Dutch-speaking graduate students, enabled the creation of subtitles to facilitate a foreigner’s (my) viewing and comprehension of the lessons.

It is important to acknowledge that although data were collected by local academics familiar with the teachers’ contexts, the analyses presented here represent one cultural outsider’s attempts to understand how participants enact their roles in culturally diverse classrooms. In such circumstances, where researchers are cultural outsiders, there is a danger of inaccurate reporting due to incomplete understanding of the cultural issues underpinning participants’ beliefs and actions (Liamputtong, 2010). Such concerns invoke a need for culturally sensitive research (Tillman, 2002, p.6), whereby researchers accept and maintain the “cultural integrity of the participants and other members of the community” by privileging the voice of the cultural group over more general and frequently dominant theories. Moreover, culturally sensitive research aims to produce culturally informed theory by drawing on participants’ culturally located perspectives on the phenomena under scrutiny. The manner in which data were collected, involving academics based at the partner universities and local teachers well known to them, was managed with appropriate regard to cultural sensitivity, as were the analyses described below.

All videos, with and without subtitles, were viewed by me several times and a narrative for each lesson constructed in which participants’ actions and utterances were recorded in as much detail, and with as little interpretation, as possible. Where they were available, narratives were developed alongside transcripts in such a way that sections of dialogue were annotated with additional details relating to participants’ actions. Where transcripts were unavailable, the narrative comprised my attempt to describe what was seen. All narratives ran to several pages of prose. These narratives were read against repeated viewings and increasingly refined. Eventually, having viewed and reviewed every lesson several times, a tentative understanding emerged with respect to the key elements of each teacher’s conceptualisation and presentation of linear equations. Throughout this process, and conscious of the need to achieve culturally sensitive analyses, one was conscious of the need to avoid evaluation of teachers’ actions alongside the desire, in accordance with conventional case study practice, to provide as thick a description of events as possible. The following reflects these ambitions.
RESULTS

The teachers were between 29 and 33 years of age with between six and eight years’ teaching experience. Each was a graduate of the project partner university and was known locally to have a commitment to both continuing professional development and school-based teacher education mentoring. All three teachers, two women, Pauline in Flanders and Emese in Hungary and one man, Sami in Finland, worked in unremarkable schools in provincial cities that were homes to the partner universities.

The analyses indicated that all three sequences comprised four phases that I have come to call, definition, activation, exposition and consolidation phases. In general, the definition phase introduced students to the notion of an equation and, either implicitly or explicitly, presented a definition. The activation phase alerted students to or revisited intuitive procedures for solving arithmetic equations. The exposition phase, through an initial presentation of an algebraic equation, exposed the inadequacies of intuitive approaches and warranted the introduction of the balance. Lastly, the consolidation phase enabled students to exploit their newly acquired skills. In the following each phases is discussed against, due to space limitations, a selection of the available evidence.

The definition phase

Lasting up to one lesson, this phase saw Sami in Finland, Emese in Hungary and Pauline in Flanders introducing and defining the topic.

Finland

The first Finnish lesson found Sami writing on the board that an equation was “two expressions denoted as being of equal magnitude”. He then wrote six “sentences”, as he called them, on the board: 5, x - 1, x = 3, 5 + 3 = 7, 3x – 1 = 4, x^2 = 8, before asking students to decide which were equations and which were not. Through constant reference to the definition, the “sentences” were categorised. Those accepted as equations were then discussed from the perspective of truth and a classification emerged that equations could be conditionally true, always true or always false.

Hungary

Emese exploited several open sentences to revisit the role of the basic set in determining a statement’s validity. This was followed by a discussion through which an equation was defined as comprising two expressions connected by an equals sign. She asserted, through her questioning, that equations may or may not contain variables or unknowns depending on circumstances and that they were always true, sometimes true or never true. Finally she operationalised her definition through an exercise in which three open sentences, 5 - 3 = 8, 5 - 3 > 6 and 0.2 = 7, were solved in relation to the basic set -3 ≤ ≤ 3.
Flanders

Pauline posed a problem involving characters from the cartoon series, the Simpsons: if Bart, Lisa and Maggie, are 7, 5 and 0 years old respectively and their mother, Marj, is 34, in how many years would the sum of the children's ages equal their mother's? She drew a two-rowed table before completing, collaboratively, the first three columns. Individual completion of the remaining columns was followed by a discussion leading to the solution of 11 years. Pauline then discursively introduced the unknown and the equation, $34 + x = 12 + 3x$, appeared, after which she sketched the straight line graph for each row in the table to highlight the intersection.

Summarising, Sami and Emese presented explicit definitions, although Sami’s, which was presented both orally and in writing, involved no student input. His definition and the tripartite classification were operationalised collectively by means of his six sentences, while Emese’s definition was discursively derived and operationalised. Her use of inequalities provided a more general entry to equation solving and facilitated an awareness of the three categorisations. Pauline exploited a *realistically*-derived equation to define, implicitly, both equation and equation solving. The latter was achieved with reference to both the table and graphs. She made no allusion to equation types.

The activation phase

This second, activational, phase saw all three teachers, as preparation for their main presentations, activate material covered earlier in their students’ careers to both contextualise and facilitate the material that followed.

Finland

Sami began his second lesson by asking for a conditional equation. One student suggested $x + 5 = 2$, with a second offering -3 as a solution. Sami then introduced $x/8 + 1 = 4$ and invited mental solutions. After a minute, despite a student offering a correct solution, he demonstrated a *covering up* method and, through closed questions and his board sponge, confirmed 24 as the solution.

Hungary

Emese began by posing oral problems like, “Kala is twice as old as her sister; the sum of their ages is 24, how old are they?” Each was solved individually before solutions were shared. Next, the class was split into four groups with each given a superficially different word problem for translating into an equation. One group's problem was: “Some friends went on a trip. The first day they covered just 2km. The second day they covered 2/10 of the remaining journey. If they covered 6km on the second day, how long was their journey?” After several minutes the group representative explained how its equation had been derived and wrote $0.2(x - 2) = 6$ on the board. Lastly, a volunteer, exploiting a *thinking backwards* strategy, obtained a solution of 32, which Emese checked against the text of each problem.

Flanders
Pauline modelled, through discussion, several sketches and an introduction to the balance, an analytical solution to $x + 7 = 9$, which was then summarised symbolically before $x - 2 = 10$, $3x = 8$ and $x/3 = 7$ were managed in the same way. This was followed by her summarising the relationship between each of her four exemplars and their respective formalisations. For example, in relation to $x + 7 = 9$ she wrote $a = b \Rightarrow a + c = b + c$. The lesson ended with her setting a homework whereby solutions to equations like $x - 3 = 10$, $200 - x = 20$ were placed in a crossword grid. The following lesson answers were shared with particular attention being paid to $3/2x = 30$ and how division by $3/2$ was equivalent to multiplying by its inverse.

Summarising, Sami invited his students to solve intuitively two equations with the unknown on one side and used the latter to introduce the cover up method that he never again mentioned. Emese exploited various word problems; initially to revisit the processes of undoing and latterly to derive arithmetic equations from realistic contexts and solve them with a thinking backwards strategy. Pauline privileged an explicit revision of arithmetical structures and their role in the solution of less straightforward equations. In so doing, she made an explicit reference to the balance.

**The exposition phase**

All three teachers began their formal exposition by presenting their students with an algebraic equation, seemingly in the knowledge that intuitive methods would fail.

**Finland**

Sami wrote $5x + 3 = 2x - 8$ and invited solutions. Once it became clear that this was too difficult, spoke of balance scales and how the same operation applied to both sides would retain the balance. Throughout he used outstretched arms to demonstrate the effect of different actions on the scales while commenting that “an equation is like scales... in principle, if you have it in balance, the equation is true”. Returning to the equation, he asked what could be subtracted from both sides of the equation. Someone suggested $x$ and Sami, without comment, wrote $4x + 3 = x - 8$. Another volunteer suggested subtracting $2x$, at which point Sami wrote, with little student input:

\[
\begin{align*}
5x + 3 & = 2x - 8 & \text{├ -2x} \\
3x + 3 & = -8 & \text{├ -3} \\
3x & = -11 & \text{├} \\
\end{align*}
\]

After some student uncertainty with regard to the next step, Sami, having asserted that they should divide by three as division is the opposite of multiplication, led the class to the solution $x = -11/3$. Lastly, individual seatwork was set from a text book.

**Hungary**

Emese began her second lesson with a word problem, “On two consecutive days the same weight of potatoes was delivered to the school's kitchen. On the first day 3 large bags and 2 bags of 10kg were delivered. On the second day 2 large bags and 7 bags
of 10kg were delivered. If the weight of each large bag was the same, what weight of potatoes was in the large bag?" Soon a volunteer wrote $3x + 20 = 2x + 70$. Then, having established that intuitive strategies were now insufficient, Emese drew a picture of a scale balance with the various bags represented on both sides. Drawing on a student’s suggestion Emese erased two small bags from each side, leaving a representation of $3x = 2x + 5$. Next she erased two large bags from each side to show one large bag balancing 5 small. Then, in response to her request, students volunteered sufficient for her to write alongside her drawings:

$$
\begin{align*}
3x + 20 &= 2x + 70 \\
3x &= 2x + 50 \\
x &= 50 \text{ kg}
\end{align*}
$$

Finally, Emese reminded her class of the importance of checking and did so.

Flanders

Midway through her second lesson, Pauline wrote $6(x - 5) - 8 = x - 3$ on the board and began a formal treatment in which the algebra, including actions, was written on the left side of the board and justificatory annotations on the right. Throughout the process, which lasted more than twenty minutes, Pauline questioned continuously. Space prevents a detailed account, although what follows represents a fragment of what was written.

$$
6(x - 5) - 8 = x - 3 \quad (1) \text{ Eliminate brackets}
$$

At this point, Pauline drew from her students notions of associativity and commutativity before settling on distributivity as the warrant for what she was about to do.

$$
6x - 30 - 8 = x - 3 \quad (2) \text{ Calculate if possible}
$$

Eventually, after obtaining a solution and discussing its uniqueness, Pauline undertook a check.

Summarising, all three teachers presented equations with the unknown on both sides with, it seemed, the intention of creating contexts in which intuitive approaches could not be exploited. All three teachers invoked the balance as an underlying principle although the extent to which it was sustained varied. Sami, having introduced the balance, made little use of it during his rather directed exposition. Moreover, despite an implicit acceptance of his first student’s subtraction of $x$, his subsequent actions indicated that he had a clear view as to what was acceptable. His solution was annotated conventionally although he invited no student input into its introduction. Emese exploited a realistic word problem to warrant the construction of her equation. She sustained the balance throughout her presentation, made explicit the relationship between her sketches and the symbolic representation and questioned her students constantly. Pauline offered the most complex of equations, deliberately provoking a
frisson of excitement in her students. Her solution process, which was driven by many questions, was very formal and invoked a number of concepts studied earlier to highlight inter-topic and structural links. Both Sami and Pauline operated in exclusively mathematical worlds although it was Pauline and Emese who included checks at the conclusions of their expositions.

The consolidation phase

The fourth phase, lasting two or three lessons, provided various opportunities for students to consolidate earlier learning and further develop both conceptual and procedural equations-related understanding. All three teachers set increasingly complex exercises, all involving algebraic equations incorporating brackets and both negative and fractional coefficients. Sami and Pauline located all their exercises within mathematics-only worlds while Emese integrated realistic word problems. Sami and Emese derived additional insights from the tasks set, while Sami introduced the change the side change the sign rule and, essentially, prescribed a preferred approach. Emese invited multiple solution strategies, discussed notions of efficiency and elegance, and constantly solutions. Pauline included a test. The manner in which tasks were completed and solution shared varied with Sami and Pauline sharing solutions after several problems had been completed while Emese always shared solutions after each problem had been solved individually.

DISCUSSION

All three sequences shared common structural - definition, activation, expositions and consolidation - characteristics. Such similarity is unsurprising as comparative studies that adopt broad and inclusive variables tend to find similarity rather than difference; as in LeTendre, Baker, Akiba, Goesling & Wiseman’s (2001) analysis of teachers’ self reported use of core instructional practices that included, for example, seatwork and whole class instruction. In other words, such broad categories tend to be inclusive and reflect patterns of instruction commonly found across cultures.

However, within these macro-level similarities were important similarities and differences. With respect to similarities, all three teachers offered definitions, either explicitly or implicitly, which were operationalised through problems and exercises. All three, having activated students’ knowledge and skills, provoked analytical solution methods by posing an algebraic equation that could not be solved by intuitive methods. All, as is discussed below, based their expositions on the balance, and all three offered extensive opportunities for consolidation that incorporated expectations of students’ managing brackets and different forms of coefficient alongside particular privileged additional insights.

In respect of differences, several issues of interest emerged. Despite similarities with respect to the balance, the manner of its introduction and maintenance varied greatly. Pauline offered only a scant allusion; Sami enacted bodily its characteristics but made no further reference once the first expository example had been solved. Emese,
through bodily enactment and sketches, made explicit the link between the embodied and symbolic forms of equation, a link sustained through several examples. Interestingly, and confounding Pirie and Martin’s (1997) scepticism, not only did all students appear familiar with the balance but also, once introduced, negatives. The examples and exercises exploited by Sami were located entirely within a world of mathematics. Pauline, having kick-started the topic with a single word problem, behaved similarly. Interestingly, while the tasks set by Sami and Pauline were generally complex, task difficulty was so teacher-managed that students experienced few teacher-independent opportunities to engage with non-routine problems. In this respect, the data suggest that both teachers had been slower than systemically desired to incorporate problem solving into their repertoires, whether in Flanders (Verschaffel, De Corte & Borghart, 1997) or Finland (Pehkonen, 2009). Emese, on the other hand, exploited both mathematical and word problems throughout and, in accordance with earlier studies, provided her students with constant opportunities to mathematise and solve text problems (Andrews, 2003). With regard to classroom norms, Emese engaged her students in collective activity focused on students’ awareness and acquisition of diversity of mathematical thinking. Pauline had clear objectives that were explicitly addressed by means of extensive but tightly focused bouts of public questioning. Sami, having operationalised his definition, exploited extensive bouts of teacher telling, interspersed with exercises, from which students were expected to infer meaning. In conclusion, although space prevents a detailed summary, such teacher behaviours, whether similar to or different from those found elsewhere, are likely to reflect characteristic patterns indicative of a national mathematics teaching script.

REFERENCES


