COMPARING THE CONSTRUCTION OF MATHEMATICAL KNOWLEDGE BETWEEN LOW-ACHIEVING AND HIGH-ACHIEVING STUDENTS – A CASE STUDY

Ingolf Schäfer*, Alexandra Winkler**

*Georg-August-University of Goettingen, **University of Bremen

The case study reported in this paper investigates whether there is a difference in the way high-achieving and low-achieving students construct mathematical knowledge and, if there is, how it might look. Furthermore, it investigates possible differences in metacognitive actions between these groups. For this, we study how pairs of high-achieving and pairs of low-achieving students deal with a problem-solving task about the divisibility of sums, using the theory of abstraction in context and a category scheme for metacognitive activities. This paper is part of a larger ongoing project that compares knowledge construction of high-achieving and low-achieving students using different tasks.

INTRODUCTION

When it comes to understanding why some students are low-achieving in mathematics there are many different approaches to the problem, including a focus on basic arithmetical difficulties often discussed in the context of dyscalculia (cf. Moser Opitz (2007) for an overview), emotional aspects like, in the extreme case, math anxiety (Ashcraft & Moore, 2009) or motivational aspects like self-esteem (Pendlington, 2006). Research also indicates metacognitive actions to be very influential in mathematics achievement (Cohors-Fresenborg et al., 2010; Wang, Haertel & Walberg, 1993). Other factors like social class or cultural background play an important role, too (Cooper & Dunne, 2000).

We restrict ourselves to looking at the process of knowledge construction while separately taking metacognitive actions into account as a complementary view. This approach is done because it is an open question whether there are structural differences in the way low-achievers and high-achievers construct mathematical knowledge and if there are whether this is mainly due to differences in metacognition as indicated in (Cohors-Fresenborg et al., 2010).

THEORETICAL BACKGROUND

Abstraction in Context

The theory of abstraction in context (Hershkowitz, Dreyfus & Schwarz, 2001; Dreyfus, Hershkowitz & Schwarz, 2001), rooted within activity theory, is a model for the process of knowledge construction that has been applied to low-achieving students (Schäfer, 2009) and processes of knowledge construction that were only partially correct (Ron, Hershowitz & Schwarz, 2006). In these studies the main benefit over other theories concerning epistemic processes is that the epistemic
actions defined below are observable actions – usually verbal – that allow insight into the internal process of abstraction.

Defining abstraction as “an activity of vertically reorganising previously constructed mathematical knowledge into a new structure”, Dreyfus et al. (2001) propose that the process of abstraction consists of three phases. In the first phase a need for a new construct arises, followed by a sequence of epistemic actions in an actual construction phase. Finally there can be a phase of consolidation of the construct.

In the construction phase three epistemic actions can be found. Recognising existing mathematical structures, building-with those structures, e.g., combining recognised artefacts to justify a particular claim, and constructing a new structure. These epistemic actions are nested, i.e., constructing incorporates building-with and recognising actions, and building-with incorporates recognising actions.

**Layers of relation to objects**

One of the directions in which activity theory has been developed is given by Oerter’s (1982) theory of action, which we will use to refine the three epistemic actions. Oerter uses the notion of action as the foundation of this theory and postulates that any interplay between individual and environment is only possible through actions. These actions, whether they are mental actions or physical actions, are always done with respect to some object, which maybe physical or mental. An individual’s relationship to objects may only be changed by performing actions.

Oerter describes three different layers of relation to an object, which we summarise briefly in the following table.

<table>
<thead>
<tr>
<th>Layer</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Singular</td>
<td>Object is only existent in the course of the action</td>
</tr>
<tr>
<td>Contextual</td>
<td>Object is discernible, but only inside a specific context of use</td>
</tr>
<tr>
<td>Formal</td>
<td>The formal structure of the object without its use</td>
</tr>
</tbody>
</table>

Table 1: Oerter’s layers of object relations (Oerter, 1982, p.114)

The singular layer is defined as the layer where subject and object cannot be distinguished by actions. Oerter gives the example of a child that uses a stick as a “sword” in playing knights. If the object “sword” is not existent anymore when the child has finished playing, then the object is only existent in the course of the action. If this playing knights is repeated more often and done with other children, the “sword” will exist beyond the actual situation, but still inside the contextual layer of play. The reader should be warned that the notion of contextual layer refers to similar situations and is different from the context in the sense of abstraction in context which includes personal history or the classroom setting for example. Thus, we always state explicitly which notion of context is meant in the following. The abstract layer is the equivalence class of the all object uses which means that only the abstract
formal structure of the object is left, e.g. the mathematical definition of a triangle is located here. For further details on Oerter’s framework and the interplay with the theory of abstraction in context we refer to the considerations in (Schäfer, 2009).

### Metacognition

The concept of metacognition, introduced by Flavell (1976), refers to the knowledge, active monitoring and controlling by the individual of her or his cognitive activities. As is shown by Schneider and Artelt’s (2010) recent overview of research in mathematics education and psychology on metacognition the concept has developed much since then. We follow the approach of Osnabrück school (Cohors-Fresenborg et al., 2010; Cohors-Fresenborg & Kaune, 2007) and study the concrete metacognitive actions of participants in mathematics learning processes. They see metacognitive actions as a triple with the components *planning, monitoring* and *reflecting*. Planning refers to actions like organising or anticipating, monitoring includes regulating, verifying and checking, and reflecting comprises evaluating, assessing and judging. Cohors-Fresenborg & Kaune (2007) suggest the additional component of discursive actions for the analysis, but we restrict ourselves to the three principal components. Based on these components Cohors-Freseborg & Kaune (2007) empirically developed a categorisation scheme for metacognitive and discourse actions. Although this scheme was developed for classroom situations in which algebraic questions are discussed on the basis of a discursive classroom culture, it is applicable to our situation as well, since the metacognitive actions themselves should not differ as long as the setting gives room for discourse.

We choose this framework for our metacognitive analysis, because it relies on an action perspective that is very fitting with Oerter’s approach to express everything in terms of actions.

### On the interplay of metacognition and abstraction in context

It is not our aim to have a unified theory of abstraction in context with epistemic and metacognitive actions, because some metacognitive actions can be part of the epistemic actions and the need. We would rather see our approach as looking at the same problem with different lenses in order to contrast and compare the findings with different theoretical perspectives (Prediger, Bikner-Ahsbahs & Arzarello, 2008).

### Working Environment: Divisibility of sums

The main task described below is concerned with the question whether, in a set of \( k \) natural numbers, there can always be found \( n \) numbers, such that their sum is divisible by \( n \). This kind of question has been used in problem solving tasks for some time now, e.g. it was the problem of the week no. 80 from Harvard Physics department (2004), which also contains a general solution with proof.

Depending on the numbers \( n \) and \( k \) the solutions of the problem can be very different. We restrict to the two cases which we used. Case 1 is \( k=13 \) and \( n=4 \), which was given to the low-achievers, case 2 for the high-achievers was \( k=17 \) and \( n=5 \). Bardy (2007,
p.72-91) used the same problem for a case study [1] on gifted students in primary school.

In both cases the fundamental insight required is that it does not matter which concrete number is given in the set of \( k \) numbers, but only the remainders matter. After restricting to the remainders each case is solved by arguing how many numbers there have to be in each remainder class. A priori we expect to find the constructs “residue class” and “Dirichlet’s box principle”.

It may seem that case 1 is more complicated because recognising the different remainder classes seems more obvious with respect to five, but that does not take into account the effort of handling an additional number in the subset. On the other hand in case 2 the combinatorics for the residue classes are more complicated and involve a case-by-case analysis.

**The notions “low-achieving” and “high-achieving”**

The school system in Bremen comprises the Gymnasium and two forms of secondary school (Realschule and Hauptschule). Each type has its own curricula, final exams, lessons per week and years to exam. While 15-year old students in the Gymnasium perform above the international average in tests like PISA 2003, the students in the Hauptschule perform below average (with over 50% at risk). The difference in performance between Hauptschule and Gymnasium students is equivalent to a difference in 3-4 grades on the average (PISA Konsortium Deutschland, 2005).

For our purpose we state that those children in Hauptschule are low-achieving whose achievement in school is significantly below their peers and who have been identified by school tests as in need of additional support in mathematics. On the other hand, we state that students in Gymnasium who achieve significantly better in mathematics than their peers and who successfully take part in mathematics problem solving competitions are high-achieving.

**RESEARCH QUESTIONS**

We are interested in answering the following questions:

1. Are there different patterns in the use of epistemic actions between high-achievers and low-achievers?

2. Are there different patterns in the use of metacognitive actions between high-achievers and low-achievers?

**METHODOLOGY AND DESIGN**

The students are presented the task on paper in form of a dialog between two children. One fictional child named Kathy says she had found out about an interesting problem in a riddle magazine and goes on (translation of German original for the low-achievers; the high-achievers had the same text with the parameters 5 and 17 and the different example sequence (22, 7, 2, 4, 6, 6, 9, 18, 6, 12, 17, 6, 11, 6, 20, 5, 16)).
Kathy: You take 13 natural numbers, e.g., 7, 2, 4, 5, 9, 14, 5, 10, 1, 5, 11, 3.

Lars: Ok, I understood. And what is so interesting about them?

Kathy: Within those 13 numbers you can always find 4 numbers, whose sum is divisible by 4!

Task:
1. Verify Kathy’s claim in her example. Find as many subsequences as possible of 4 numbers whose sum is divisible by 4.
2. Is it really always true what Kathy claims?

The students are asked to discuss these problems with each other and write down a solution. They are given no additional information or help, with the exception of certain prompts from a field manual that should be given to the low-achieving students at certain times to ensure that they recognise certain phenomena, e.g., one of the authors might ask the students to explicitly compare two subsequences they found in order to help them recognise the fact that it is possible to replace a number with another representative of the same equivalence class without contravening divisibility. The task was mainly chosen because it does not depend on special knowledge other than basic arithmetic skills (other tasks are planned in future). No such prompts are provided for the high-achievers. The whole process is then videotaped and transcribed. The data is analysed in two separate turns: Each utterance in the transcript is coded according to the coding guidelines for the metacognitive actions as described in Cohors-Fresenborg & Kaune (2007). Separately, the analysis according to the RBC-model was done in a sequence analysis using an interpretative approach to the text utilising the theory of speech acts (cf. Bikner-Ahsbahs, 2008).

FINDINGS

Our case study was conducted with 2 pairs of high-achieving grade 6 students in Gymnasium and 2 pairs of low-achieving grade 9 students in Hauptschule meeting the definition. The difference in grades is supposed to roughly compensate for the difference between the school types as explained above. Due to space limitations we can only give a short summary here.

The case of Alice and Betty

Alice and Betty [2] are in the low-achievers group. They look for subsequences for question 1 more or less by trial and error and do not make much use of additional structure besides using the sum of the subsequences for ordering and comparing, which they talk about explicitly, but use rarely. After a prompt by the interviewer they realize that certain numbers belong to the same remainder class, but it seems that they do not come to a more concrete understanding of the concept. Other than stating that they would need to find other examples, since one example is no proof and guessing that the product of the numbers in the subsequence might be involved, they do not come to arguments for question 2.
The case of Carl and Dan

Carl and Dan are also in the low-achievers group. For question 1 they struggle with the formulation of the task where Carl initially understand “subsequences of four numbers” as “four subsequences of four numbers”, which is clarified by Dan and the interviewer, but 30 minutes later Dan shows the same misunderstanding. Due to a prompt by the interviewer (“in a subsequence a 5 can be replaced by 1 without changing the divisibility of the sum”) they are able to use the concept of remainder class and Dan even finds the general definition of equivalence. Although they make some considerations on the divisibility of even and odd numbers, they do not achieve a proof by themselves.

The case of Eric and Fred

Eric and Frank are high-achievers. They work separately for most of the time while still informing each other about their respective findings and apply some strategies in finding examples including Eric doing a systematic case-by-case testing for subsequences of length 4, which he stops at some point. Fred tries to assemble a counter example and realises that it is possible to restrict the numbers from 1 till 9 for the sequence. Eric builds on this and introduces remainder classes. This restriction, in combination with Fred’s combinatorial considerations on the way to construct a counter example, leads to the proof given above in the mathematical considerations.

The case of Greg and Hank

Greg and Hank are high-achieving students. In answering question one they use many different strategies to find subsequences, most prominently a replacement strategy where they start with a given subsequence and replace numbers in it such that the sum remains a multiple of 5. They explicitly call these “rules” and give example subsequences with rules to modify them. Working with those rules they find the concept of remainder classes, writing down the first ten elements of each class. Restricting to the representatives 1 to 5 they give the proof outlined in the mathematical considerations.

Similaries and differences in the RBC-model

For all groups we noticed many instances of recognising and building-with actions, but constructing mainly occured among the high-achievers. Regarding the quantity of epistemic actions we note that the high-achievers performed a greater number of such actions in a fixed time span than the low-achievers. But one should keep in mind that although we tried to compensate by using smaller numbers in the task for the low-achievers, they spent much time on arithmetical actions, mainly adding the numbers, leaving less time for epistemic actions. Most of the epistemic actions happen at the situational or contextual layer in all groups and we could reconstruct three common contexts in the sense of Oerter’s theory of action among the groups:

1. **Arithmetics**, in which the sums of the subsequences are calculated and some important properties can be recognised, for instance the invariance of the
divisibility under replacement of a number by another representative of the same remainder class

2. *Combinatorics*, in which possible subsequences are explored, e.g., how many pairs of numbers sum up to 12, or how many numbers have to be at least in each remainder class.

3. *Understanding of the task*. Some recognition actions are about what the task is and what it is not, i.e., how many subsequences must be found, how to deal with duplicate numbers etc. Since these actions are on-going for a longer period of time (a couple of minutes) and in all the groups, we consider them contextual rather than situational.

The low-achievers spent most of their time on situational epistemic actions or epistemic actions on the contextual layer regarding arithmetics or understanding the task and showed very few formal actions, the high-achievers showed more formal actions and also were less occupied with the arithmetical contextual layer.

We want to focus on a specific example of epistemic action which illustrates the difference between low-achievers and high-achievers. At some point the students in every group had found a subsequence of \( n \) identical numbers, but this did not lead to the same kind of epistemic actions. When confronted with the specific case of the number six while dealing with question 1, Greg and Hank immediately contextually recognise that this is \( n \) times six, i.e., multiplication and thus, divisible by \( n \). Later when they deal with question 2 they recognise this case on the formal layer.

Hank: So, if you got five of one [number], you can divide it by five in any case.

Eric and Fred deal with the example of 6, 6, 6, 6, 6 in no special way in question 1. They just treated it as an example like the others, but in dealing with question two, they recognized, on the formal layer, that five times an identical number is divisible by five. For the low-achievers this piece of knowledge has probably not previously been constructed in a way that they could recognise it. In the case of Carl and Dan, they constructed multiple examples of sequences of identical number and Dan calculated the sum of four of them to check, whether they were divisible by four. This means they were building-with the idea that the sum of four identical numbers is divisible by four contextually and seemed to be unable to recognise the structure. Alice and Betty did not find the example of 5, 5, 5, 5 in working on the first question. When they worked on the second question they followed a prompt by the interviewer to look at sequences containing only the numbers 1 to 4 first. Betty suggested looking at sequences containing only one number, for example the number 1.

Alice: Yes, then I can take – then – I can take four times one. Yes, four.

Then they tried four times two and four times three by calculating the sum and checking whether it is a multiple of four. In the case of four times four she said

Alice: … and four times four is – works anyway.
which may indicate that she had recognised some structure here. But as the episode shows, they did not recognise on the formal layer, but rather recognised the divisibility of each result in the first three examples in the contextual layer.

There was also a difference between high-achievers and low-achievers regarding falsely recognising a mathematical property. This happened for all groups, but there was a difference in the way it evolved. In the case of the high-achievers they dropped those ideas when they found a counterexample, the low-achievers were usually willing to keep them for at least another counterexample.

In summary, we uncovered several differences regarding Oerter’s layers between the groups, e.g., the high-achievers may “easily” recognise a formal property, while low-achievers have to built-with it contextually. We were unable to find more systematic patterns in our cases.

**Similarities and differences in the metacognitive actions**

All four groups used many different metacognitive actions in all of the three components, but there were differences in the use of certain subcategories. No actions occurred in the subcategories related to different representations, which may have been due to the fact that all groups solely used the representation by numbers. In the following the letter and the number in curly brackets refers to the metacognitive actions in the scheme of Cohors-Fresenborg and Kaune (2007).

Regarding the planning component there were few differences between the two groups, but in the monitoring and reflecting component, we saw differences. The high-achievers reflected on the notions used \{R1\} and changed their point of view numerous times \{R4\}, the low achievers did not reflect on the notions and changed the point of view only in one case. But even for those metacognitive actions that were common to both groups there were differences in quality and use. Those differences can be described by three themes, which we illustrate by means of specific examples.

One of the themes is the *broadness of the content* to which the metacognitive strategies are applied. For example in the monitoring of deficits in understanding or planning \{M5, M6\} the low-achievers only applied this to each others’ direct actions and utterances or in understanding the questions.

Dan: What does she claim then? [regarding Kathy in question 1]

In contrast, the high-achievers also monitored with regard to the purpose of certain steps or intermediate results.

Eric: What do we get out of it, when we know how many groups there are?

The second theme concerns *the interplay of metacognitive actions*. All groups actively searched for divisible subsequences and monitored their actions with respect to the goals \{M7\}, but the low-achievers mainly utilised calculating monitoring strategies for this \{M1\}, while the high-achievers planned in advance \{P2, P3\}, e.g.,
Eric fixed three numbers, stated possible numbers to be added in general and controlled which of these numbers occurred in the example and were thus allowed.

The last theme relates to differences in the *perseverance of metacognitive actions*. While high-achievers and low-achievers alike monitored their actions with respect to content and goals {M7}, the low-achievers seemed unable to maintain the level of monitoring all the time, e.g., both groups of low-achievers at first calculated the total sum of a sequence of 13 identical numbers, when working on question 2, instead of calculating the sum of only 4 numbers. The high-achievers also studied the example of sequences where each number was identical, but correctly use only partial sums.

In summary, our cases indicated a difference between the groups with regard to metacognition, but more about the quality than about simple occurrences.

**SUMMARY AND OUTLOOK**

The above analysis indicates that it is worthwhile comparing the processes of knowledge construction between low-achievers and high-achievers. In the cases studied there seem to be great differences already in the recognising actions, which then in turn affect the possibility of building-with and constructing due to the nested nature of the epistemic actions. The role of contextual actions and the different themes in using metacognitive actions also seem to be influential. But, as was mentioned above, the arithmetical side of the problem was considerately more time-consuming and therefore probably harder for the low-achievers. Since this may account for some of the differences between the groups regarding the process of knowledge construction, we are in the middle of performing two additional tasks – one related to graph theory and another to geometry – with the students to see how much is related to these arithmetical problems.

**NOTES**

1. The question and the transcript itself is not part in the book. We thank Peter Bardy for sharing them with us.

2. The names of the participating children have been replaced by arbitrary chosen names.

**REFERENCES**


