In this paper, the main components of a conceptual framework for the study of visualization processes at University are presented along with a discussion of examples of empirical data relating to the concept of integral: (1) the coordination of registers of representation for explaining some of the students' difficulties in the understanding and learning of mathematical concepts (the integral); (2) characteristics of visualization (in Calculus): it is related to the use of the graphic register in coordination with other representations and accompanied by a global apprehension and; (3) the use of the graphic register and the higher cognitive difficulty of visual methods. In the construction of this conceptual framework both cognitive and sociocultural-affective aspects of visualization have been taken into account.

Key-words: Visualization, Representations, University Level, Integral.

INTRODUCTION

This paper is the consequence of a study conducted in 2008/2009 with a first year group of students at the Universidad Complutense de Madrid (UCM). Its main aim was to improve the teaching of Mathematical Analysis at University Level by emphasising visualization processes (Souto, 2009; Souto & Gómez-Chacón, 2009). The literature review related to this problem highlighted a large diversity of terms and theories around the notion of visualization. Two main approaches were found: (1) research in Didactics of Analysis (Mundy, 1987; González-Martín & Camacho, 2004), with a more cognitive perspective; (2) research on visualization in Mathematics Education (Arcavi, 2003; de Guzmán, 2002; Presmeg 1985, 2006; Eisenberg & Dreyfus, 1991), which from different perspectives includes sociocultural and affective issues. The empirical data obtained in this study suggested it would be useful to consider both perspectives at the same time. Within the cognitive approach, the theory of registers of semiotic representation (Duval, 1995, 1999) was useful in describing and analyzing students’ difficulties in the learning of mathematical concepts. More sociocultural and affective approaches were relevant in order to explore causes for these difficulties and to shed light on teaching processes.

Therefore, the need to build our own conceptual framework for visualization at University Level by integrating and combining ideas from both perspectives was highlighted. For the sake of brevity, however, our paper focuses the cognitive elements of this framework. Firstly, the main theoretical ideas of this conceptual framework are outlined and reviewed. Secondly, these ideas serve to analyze some
examples of our empirical data on the concept of integral. This concept has been chosen because it provides an opportunity to discuss key issues concerning visualization. Research on the concept of integral (Mundy, 1987; González-Martín & Camacho, 2004) emphasizes that during the first year of university, students use the concept of integral in a very mechanical way. This may be due to the lack of coordination of the concept of area and of integral. Thus, some research pays attention to the coordination of the graphic and algebraic registers. Furthermore, in an attempt to improve the comprehension of the concept, one noteworthy teaching recommendation is to pay explicit attention to visualization (de Guzmán, 2002; González-Martín & Camacho, 2004). Our analysis of the examples of empirical data has highlighted some challenges concerning teaching with a focus on visualization.

ABOUT THE CONCEPTUAL FRAMEWORK

The theoretical framework of the cognitive theory of the registers of semiotic representation (Duval, 1995, 1999) is useful in order to describe students’ difficulties in the understanding and learning of mathematical concepts. However, research on visualization enables us to go beyond the semiotic approach and to incorporate more affective and sociocultural perspectives. Among them we highlight the role of intuition in mathematical reasoning (de Guzmán, 2002; Arcavi, 2003), individual differences in the preference to visualize (Pensmef 1985, 2006); and reasons for a reluctance to visualize (Eisenberg & Dreyfus, 1991).

Understanding and learning of mathematical concepts

We agree with Duval (1995, 1999) on the possibility of having direct access to mathematical objects only through their representations in the different semiotic registers. From this perspective, the understanding of a concept is built through tasks that imply the use of different systems of representation and promote the flexible coordination between representations. Therefore, learning mathematics implies “the construction of a cognitive structure by which the students can recognize the same object through different representations” (Duval, 1999: 12).

In this context, improving learning supposes to minimise difficulties, misunderstandings and mental blocks that could appear in different actions related to a register: representation, treatment and conversion (Duval, 1995). As we noted in the introduction, in our specific case - the understanding of the concept of integral- research conducted with this semiotic approach highlights as a cause of these difficulties the lack of coordination between both the graphic and algebraic registers and the predominance of the latter in the students’ answers. This leads us to pay special attention to the use of the graphical register and to visualization.

Visualization in Mathematics Education

According to Duval (Duval, 1999: 15), visualization can be produced in any register of representation as it refers to processes linked to the visual perception and then to
vision. For the aim of the study this notion is too broad; although we take into account some other characteristics of visualization pointed out by Duval. We find more useful Arcavi’s definition (2003: 217) in which visualization is limited to the use of figures, images and diagrams. Therefore, in the frame of this research, we identify visualization with the use of the graphic register. However, there are some caveats to this statement.

In the characterization of visualization in the context of problem solving, we find very useful the difference between visual and non-visual methods established by Presmeg in her research about preference to visualize (Presmeg 1985). However, we have to be cautious when using both the terminologies of Presmeg and Duval. For example, the following equivalence cannot be established: visual method (Presmeg) - use of graphic register (Duval). We must be cautious for two reasons. Firstly, when Presmeg (2006) talks about visual images, she includes mental images that belong to the world of mental representations, very different from Duval’s semiotic representations (Duval, 1995: 14). We agree with this inclusion of mental images in relation to visualization. Secondly, the use of the graphic register does not imply that the method is visual. Duval (1999: 14) distinguishes two types of functions for the images: the iconic and the heuristic. The latter involves a global apprehension and it is related to visualization (Duval, 1999: 14). If there is use of the graphic register but there is not global apprehension or the image performs an iconic function, it is not possible to talk about visualization. Thus, the connection should be made between visual methods (Presmeg) and this heuristic function of images (Duval).

This connection between the theories of Duval and Presmeg has an important consequence as it favours the consideration also of more affective and sociocultural issues. It enables the individual differences in the preference to visualize to be looked at and therefore asking the question: which factors are influencing these differences? In this way, Eisenberg and Dreyfus (1991) indentify three reasons to explain the reluctance of some students to visualize: “a cognitive one (visual is more difficult), a sociological one (visual is harder to teach) and one related to beliefs about the nature of mathematics (visual in not mathematical)” (1991: 30).

**PARTICIPANTS AND DATA COLLECTION**

The study was conducted with a first year group of 29 Mathematics students at UCM, 15 female and 14 male. In this first year, the students had a subject called Real Variable Analysis, in which the formal definition of the concept of integral is introduced. However, they were supposed to have learned the basic rules for integration by using primitives as well as its relation to the calculation of some areas under curves previously, in high secondary school.

For the data collection, the instruments used were a questionnaire with problems and semi-structured interviews. The questionnaire was composed of 10 non routine problems in Mathematical Analysis, some used in other research (Mundy, 1987;
works quoted in Eisenberg & Dreyfus, 1991). Most of the problems are posed in the algebraic register but they also allow a visual interpretation (Eisenberg & Dreyfus, 1991). Thus these problems allow the analysis of students’ performance with regard to the coordination of registers, and particularly the use of the graphic register. The results obtained from the questionnaire required deeper investigation into affective, cognitive and sociocultural aspects of individuals. In order to do this, 6 semi-structured interviews were conducted. They were divided into several parts: characterization of the individual, tasks about beliefs and preference of visualization, questions on questionnaire’s answers.

For the data analysis, we privileged the use of systemic networks for the questionnaire and transcriptions for the interviews. Systemic networks enable all the students’ answers to the problem questionnaire to be looked at simultaneously. In particular, this configuration (Figure 1) favours the observation of the following elements: strategies and kinds of representation used by each student [1]; frequency of use and difficulties of each register; and students’ conceptions.

ANALYSIS AND DISCUSSION OF RESULTS

Students' difficulties analyzed through the theory of registers of representation

Students’ answers to the questionnaire were analyzed using the first component of our framework, based on Duval’s theory of registers of semiotic representation. The results described below are based on the analysis of the systemic network associated with the following problem (Figure 1), but they are representative of what happened with other problems in the questionnaire.

What’s wrong in the following calculation of the integral?

\[
\int_{-1}^{1} \frac{1}{x^2} \, dx = \left[ \frac{x^{-1}}{-1} \right]_{-1}^{1} = \frac{-1}{1} - \frac{-1}{-1} = -2
\]

Firstly, the choice of representation and register is very important for solving the problem successfully. These decisions are directly related to the conception used for the integral concept. All the students who gave valid answers were either focused on the function (global properties as continuity or asymptotes; or local in \(x = 0\)) or contemplated the integral as a product (area under a curve). This led two of the students to use the graphic register. However, most of them (22 students) interpreted the integral as a process (calculation of primitives and Barrow Rule or integration as inverse process of derivation) which led all of them to the use of the algebraic register. In this case, the students did not achieve complete understanding about what was happening and then either repeated the same calculation or made some errors (using a different primitive, interchanging the signs when applying the Barrow Rule, considering the constant of integration, miscalculations).

Secondly, we examine the way in which representations are used. The data show that the initial kind of representation chosen does not determine completely the success in the resolution. For example, student 4 was the only one who at the beginning focused
on the integral as the calculation of primitives, but answered successfully. This was possible because of the flexible combination of this calculation with another argument about the domain of definition of the function. Moreover, the analysis of the answers highlights how the coordination of registers led to a better understanding of the problem (Figure 1, answers 28 and 29). But the important thing is that the mobilisation of both registers is not made mechanically. It should be accompanied by reflection; otherwise it could result in some errors (see in Figure 1, answers 19, 27).

**STUDENTS’ ANSWERS**

- As the function is not continuous, Barrow Rule can’t be applied. (1)
- Recalculate the primitive (with +C) and The primitive of 1/x is calculated correctly but the integral is not well defined, as x = 0 (7)
- The f is not defined in 0 (14)
- In x=0, f goes to infinite (17)
- In x = 0 the function goes to infinite. The integral can’t be calculated in that way for that function. (28)
- The 2nd = is wrong (without justification). The function is drawn and it is concluded that it cannot be negative. (22)
- The same steps are repeated (20)
- Other primitives are used (13)
- Other errors (26)
- Comments (3)
- Problems with the integral (10)
- It’s correct (3)
- Without justification (6)
- I cannot integrate. For me, it would be right. (9)
- Problems with variables of integration (11)
- Others examples of primitives (12)
- Problems with primitives such as ∫ x^n dx = x^(n+1)/(n+1) when n is positive. This method cannot be applied to negative exponents. And he puts as an example n = 1. (21)

**CONCESSIONS**

- Continuity. Function as part of the hypothesis of a Theorem (4)
- Function as application that gives one value to another. (15)
- Function depending on one variable. Idea of limit, asintotes. (18)
- Integral as an area under a curve. (23)
- Integral as a calculation of primitives (19)
- Integral as an area under a curve. (27)
- Visual: Integral as an area under a curve. (29)
- Definite Int. Barrow’s Rule (5)

**Figure 1: Systemic network associated to the problem**

Thus, our results are coherent with previous research (Mundy, 1987; González-Martín & Camacho, 2004) described in the introduction. From a didactic point of view, the use of different kinds of representations and registers seems to be essential. But, how to promote their flexible coordination when teaching? And, is this coordination enough for the students to achieve comprehension of the concepts? Our analyses suggest that the answer to the latter question is negative.
Essential for visualization: the global apprehension

Global apprehension of images is required together with the coordination of registers. However, some students do not go further than having a local apprehension and cannot see the relevant global organization (Duval, 1999: 14). Our analyses coincide with these ideas as we try to show with the description of the following episode from the interviews.

Figure 2: Heading of the task based on Young’s Inequality

The episode concerns Young’s Inequality (Figure 2) and the interpretation of a graphic and its subsequent connection with the heading of the theorem is asked. Silvia is the name of the student chosen for the interview. She was selected because her responses to the questionnaire showed some preference for the graphic register, but she did not answer satisfactory any problem. The interview enabled us to go deeper in her difficulties with visualization.

At first, only the image was shown. Silvia detected isolated elements and even made some references to the integral as an area. Later, we showed her the statement of the inequality. She assumed the relation to the image, but it did not seem to be clear for her. She frequently requested help by asking questions. Afterwards the following conversation took place. In order to be able to continue with the interview, support was given to help her to identify correctly all the elements in the image with those in the statement. In spite of this, there was not any “aha!” moment of understanding.

Interviewer: OK, what kind of explanations would you need with the drawing? Have you understood it completely, the drawing?

Silvia: Um… Well…[…]

Interviewer: OK, this ab, the rectangle] is equal or less than this integral, the one which is in the drawing?

Silvia: Well, it’ll be this, from 0 to a (she points with the finger to an interval over the x-axis). This one, the $S$’s.

Interviewer: OK, and the other?

Silvia: Well, $T$’s. (Silence, she seems pensive)

Interviewer: This is a little more difficult for you to see, isn’t it?

Silvia: Yes.[…] Well, to understand it [the theorem], with the drawing I wouldn’t understand it.
Silvia could not go beyond the mere identification of the represented units. For her, the image was only an illustration, an iconic representation that does not work as a means of visualizing the statement of Young’s Inequality. Therefore, the main conjecture for Silvia’s difficulties with visualization in this case is the lack of global apprehension. From a didactic point of view, the following challenges emerge: Is it possible to teach how to apprehend globally an image? If so, how can it be done?

**The high cognitive requirement of visual methods**

During another task in Silvia’s interview, she explained why she chooses “the way they give [in class], the definition” as follows: “I don’t know. It’s like everything is more mechanical. In the other way [visual] you have to relate, to think. […] It isn’t that I prefer it [algebraic], but it’s easier. So, instinctively, I do it”. This excerpt of the interview concerns the cognitive rationale pointed out by Eisenberg and Dreyfus (1991) for the reluctance to visualize. In order to go deeper into this issue, the students’ use of the graphic register in the answers to the questionnaire was analyzed. Taking into account the distinction between iconic/ heuristic functions performed by images (Duval, 1999) and its relation with non-visual/ visual methods (Presmeg, 1985), different kinds of methods for solving a problem using the graphic register were detected: non-visual, mixed and visual. The data collected from the following problem of the questionnaire allow us to illustrate some characteristics of each kind of argument.

\[
\text{If } f \text{ is an odd function in } [-a,a] \text{ calculate } \int_{-a}^{a} (b + (f(x))) \, dx
\]

This problem was answered by 20 students, and only 8 used the graphic register. The first kind of argument (Figure 3) appeared with more frequency (5 out of 8 students). The images appear together with the algebraic register, in which the main argument takes place. The images were employed either to try to remember the definition of odd function or to deduce some other properties. Therefore, the image was unnecessary and it performed an iconic function. The resolution was considered to be non-visual. In fact, the example shown (Figure 3) is accompanied by an image that the student interpreted by giving an incorrect definition of odd function. In spite of this, this misunderstanding did not affect the algebraic reasoning, which is valid.

In the other two arguments, the images were interpreted as performing a heuristic function. However, they are different according to the number of conversions made between the algebraic and the graphic registers. The second argument (Figure 4) was given by two students. It has been called mixed as a first step is needed in the algebraic register, in which the additive property of integrals is applied, before converting to two graphic representations, one for each integral. As an informal conversation with the student who gave the answer in Figure 4 clarified, this conversion allowed him to calculate the value of the integrals, without doing operations, and then to come back to the algebraic register to finish the evaluation of the integral. Thus, two conversions were made (algebraic- graphic- algebraic).
The third argument (Figure 5) is completely visual since it includes at the beginning just one conversion to the graphic register in which the main argument is developed. This original answer given by only one student provides the opportunity to show how the cognitive theory of registers of semiotic representation serves to explain and to go deeper into the cognitive difficulty of visualization noted by Eisenberg and Dreyfus (1991). In order to give this visual answer, more concepts and relations than in the non-visual ones (including the pure algebraic one) must be considered simultaneously: a visual interpretation of the integral as an area and the odd functions as those symmetric on the point (0, 0); the recognition that adding a constant quantity to a function means a translation of its graphic along the y-axis (this argument is not present in the algebraic argument); finally some image treatment of that sort of “cutting” and “gluing” areas, taking into account their signs. There is also at the end an algebraic answer used for checking. It follows a linear process consistent in the succession of several operations that are made without errors, giving as result $2ab$. Obviously, in both arguments, the result is the same. However, each kind of argument leads us to see the problem in a very different way.

Therefore, from a didactic point of view, the combination of visual and non-visual arguments when teaching seems to be advisable since it provides complementary kinds of understanding. However, as it was argued in the conceptual framework and as the empirical data have shown, this should not be misinterpreted as just using the graphic register. The following challenges for teaching emerge: how to combine visual and non-visual methods in class in order to improve the understanding of the students? How can the higher difficulty of visual arguments be handled in the class?
Moreover, as we pointed out in the conceptual framework, this distinction of visual, mixed and non-visual methods allows individual differences to be looked at, and other kind of questions arise: can students’ profiles be establish attending to their preference for using these three different methods? Which are the factors influencing these individual differences? How to handle these differences when teaching?

CONCLUSION

In this paper, we aim to: (1) present the integration of the main ideas found to be relevant both in the literature and in the empirical data in a conceptual framework for visualization at University Level, mainly from a cognitive perspective; (2) show how this new tool provides insight at both levels, theoretical (as different theories complement each other) and empirical (as it offers a rich data analysis).

The theoretical framework of the cognitive theories of semiotic registers of representation (Duval, 1995, 1999) was useful in order to: (1) describe some difficulties of the students in the understanding and learning of mathematical concepts, in this case, the integral; (2) define some necessary conditions for visualization (in Calculus): it is related to the explicit or implicit use of the graphic register and to the coordination with other representations (in the same or different registers) and, as Silvia’s episode showed, the image should be necessarily accompanied with a global apprehension; (3) examine the students’ use of the graphic register and the higher cognitive difficulty of visualization argued by Eisenberg & Dreyfus (1991). Moreover, visualization is related to the heuristic function of images (Duval, 1999) which has been identified with the visual methods (Presmeg, 1985). This connection inspired us to distinguish three different kinds of methods for solving a problem using the graphic register: non-visual, mixed and visual. Although this possibility has not been exploited in this paper, this idea is important because it enables us to shift our attention to individual differences in the preference to visualize. In this way, it is possible to incorporate other perspectives present in research in Mathematics Education in the visualization that include more affective and sociocultural issues going beyond the semiotic approach.

From a didactic point of view, some challenges around a specific teaching of visualization emerge: how to use different kind of representations and registers in order to promote a flexible coordination between them? Is it possible to teach how to apprehend an image globally? How could it be done? How to combine visual, mixed and non-visual methods in class in order to improve the understanding of the students? How to manage the higher difficulty of visual methods? How to handle individual differences in the preference to visualize when teaching?

NOTES

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1. Students were labelled with a number from 1 to 29. It appears between ()in the systemic network.
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