Abstract. In this paper I draw on interviews with university mathematicians in order to discuss their comments on Year 1 students’ verbalisation skills. Analysis was conducted in the spirit of enculturative and discursive theoretical perspectives and here focuses on: the role of verbal expression to drive noticing; the importance of good command of ordinary language; the role of verbalisation as a semiotic mediator between symbolic and visual mathematical expression; and, the precision proviso for the use of ordinary language in mathematics. I conclude with the observation that discourse on verbalisation in mathematics tends to be risk-averse and that more explicit, and less potentially contradicting, pedagogical action is necessary in order to facilitate students’ acquisition of verbalisation skills.

Key Words: university mathematicians, verbalisation, enculturative and discursive perspectives, student learning.

Being able to use ordinary language in order to construct and convey mathematical meaning is an indispensable tool of the mathematical thinker. The mathematics community has always revered eloquent mathematical exposition and, at least until a collectively accepted and used symbolic mathematical language became increasingly dominant from early 19th century onwards, substantial ideas in mathematics (by luminaries such as Cauchy and Lagrange) were often conveyed in ordinary language.

Yet research on how mathematicians and their students acquire verbalisation skills is scarce or subsumed in broader studies of mathematical expression. In the early 1990s research on students’ handling of the verbal and symbolic elements of mathematics language often focused on students’ comprehension and response to mathematical texts, rather than students’ own generated verbal utterances. For example, Furinghetti and Paola (1991) discussed students’ understanding of mathematical texts in the context of difficulties with formal proof. Dee-Lucas & Larkin (1991) found that: proofs written in ordinary language resulted in better student performance than equation-based proofs on problems related to both equation and non-equational content; the presence of equations cause students to shift attention away from non-equational content; and, learners have more difficulty processing equations than verbal statements of the same content. With regard to students’ own writing some studies (for example, MacGregor, 1990) have suggested that writing sentences helps students write correct symbolic expressions.
In this paper I explore some of the characteristics that make verbalisation skills an important component of mathematical learning. I also discuss some of provisos for an appropriate and effective use of verbal expression in mathematics. To this purpose I draw on data collected for a broader study (Nardi, 2008) which aimed to elicit university mathematicians’ perspectives on the teaching and learning of mathematics at university level. In the following I introduce briefly the study and the theoretical underpinnings of the discussion I present in this paper.

A STUDY OF MATHEMATICIANS’ PEDAGOGICAL PERSPECTIVES

The study I draw on in this paper is in an area that has been gaining increasing interest in university-level mathematics education research in recent years: the study of university teachers and teaching, with a focus on the pedagogical and epistemological perspectives of university mathematicians (e.g. Jaworski, 2002). The study has its theoretical origins in several traditions of educational research such as: clinical partnerships between researchers and practitioners (Wagner, 1997); communities of practice (Wenger, 1998); Schön's (1987) reflective practice; and, Chevallard’s (1985) notion of transposition didactique. From these traditions the study has acquired the following characteristics – see (Nardi, 2008, p6-9) for an elaborate account. It is: collaborative (namely, it brings together mathematicians and researchers in mathematics education in a collective discussion of pedagogical issues); context-specific and data-grounded (namely, this discussion is conducted in the specific context of, and with data from, the participating mathematicians’ teaching experiences); non-deficit (namely, the discussion, while encouraging self-reflection and critique, does not primarily aim at the identification of problematic aspects of mathematicians’ teaching by the mathematics educators); and, non-prescriptive (namely, the discussion, while it encourages the identification of preferred and recommended practice, does not lead to explicit pedagogical prescription).

The data collected for the purpose of the study consist of focused group interviews (Wilson, 1997) with twenty-one mathematicians of varying experience and backgrounds from across the UK. Eleven interviews with groups of three to five mathematicians were conducted, lasting between two and four hours. Discussion in the interviews was triggered by Student Data Samples, namely samples of students’ written work, interview transcripts and observation protocols collected in the course of earlier studies of (overall typical in the UK) Year 1 introductory courses in Analysis / Calculus, Linear Algebra and Group Theory. See (Nardi, ibid, p9-14) for summaries of these studies. The Samples were sent to the interviewees at least a week prior to the interview with a request to read and reflect on them in preparation for the interview.

In accordance with data-grounded theory techniques (Glaser and Strauss, 1967) the approximately 250,000 words of interview transcript were initially arranged in clusters of episodes on the following six themes: four focused on student learning
students’ mathematical reasoning; in particular their conceptualisation of the necessity for proof and their enactment of various proving techniques (1); students’ mathematical expression and their attempts to mediate mathematical meaning through words, symbols and diagrams (2); students’ encounter with fundamental concepts of advanced mathematics such as Functions (3) – across Analysis, Linear Algebra and Group Theory – and Limits (4)); one focused on university-level mathematics pedagogy (5); and, one focused on the often fragile relationship between the communities of mathematics and mathematics education (6). There were approximately 80 episodes. The data I sample and discuss in this paper concern the issue of students’ verbalisation skills and collate evidence interspersed in 7 out of the 80 episodes.

The discussion here is in the light of enculturative and discursive perspectives on mathematical learning. In terms of the former (e.g., Sierpinska, 1994; Wenger, ibid), students are seen as incoming participants to the practices of the mathematics community and mathematical learning is the students’ knowledge development through communication and practice; in this process the main role of their university teachers is to introduce them to these practices.

In terms of the latter (e.g. Sfard, 2007), the learning of mathematics involves a change of discourse, where discourse is meant as a distinct form of communication that a community engages with. A discourse is made distinct by the community’s word use, visual mediators, endorsed narratives and routines. Words include mathematical terminology. Visual mediators include diagrammatic and symbolic representations of mathematical meaning. Endorsed narratives include definitions and theorems. And routines include practices such as conjecturing, proving, estimating etc.. Sfard (ibid) describes the changes of discourse involved in mathematical learning in terms of two levels: object-level (namely adding endorsed narratives, e.g. accumulating knowledge of new definitions and theorems) and meta-level (namely adding new objects, changing rules of the discursive game, changing word use etc.). Below I outline the analysis of the episodes – overall but particularly those clustered under theme (2) above – in the terms of these two perspectives.

Students’ mathematical expression – whether verbal, visual or symbolic – is expected to undergo a substantial shift, particularly in the early parts of their university studies. At least in the UK where the study was conducted, mathematical discourses in school and at university are markedly distinct. Brief examples of the differences between the two discourses involve routines such as proving (in school students are rarely, if at all, expected to provide a formal proof of a claim that they make) or the employment of certain visual mediators (in school students are not generally expected to make extensive use of formal mathematical language, symbols such as quantifiers etc.). The interviewed mathematicians paid particular attention to the tension that they see their students experiencing while undergoing this discursive shift. Below I list five characteristics that, according to the participating mathematicians, typify this tension:
• **Inconsistent Symbolisation:** students’ attempts at producing ‘acceptable’ mathematical writing result in inconsistent use of symbolic language;

• **Ambivalent Visualisation:** students’ appreciation of the role of visualisation in gaining and presenting mathematical insight is ambivalent. Their use of it is lacking in confidence or lacking altogether.

• **Undervalued Verbalisation:** students undervalue, and often avoid entirely, expressing their mathematical thoughts verbally.

• **Premature Compression:** students’ mathematical writing is typically prematurely compressed, namely ridden with gaps, leaps of logic and omissions.

• **Appearances:** students often enact their perception of the need to be mathematical (use the discursive norms of mathematical reasoning such as providing justification or proof etc.) as a need to appear mathematical (appear to make extensive use of mathematical symbols, terminology or expressions).

The last two characteristics can be seen as combined repercussions of the first three. Data substantiation of each of the five can be found in (Nardi, 2008, mainly Chapter 4). Further elaboration on relationships across all five is part of a longer paper that is currently in preparation. Here I focus on the third characteristic, **Undervalued Verbalisation**, in order to present and discuss the interviewed mathematicians’ pedagogical and epistemological views on the role of verbalisation in mathematics.

**THE MATHEMATICIANS’ CASE AND PROVISOS FOR VERBALISATION**

Across the seven episodes from which I collate data for the purpose of the discussion in this paper the interviewed mathematicians reported extensively the students’ lack of both verbal ability and appreciation for verbal expression in mathematics. At the heart of this reporting was the mathematicians’ concern that students’ inadequate appreciation for, and engagement with, verbal expression was an indication of the students’ difficulty with – and lack of awareness of their obligation for – making their thinking as transparent to the reader as possible. The mathematicians also appear concerned about the students’ lack of awareness of the benefits that come with the mastering and employment of verbalisation skills in mathematics. In what follows the discussion of the data is structured around four key issues that emerged as significant from the interviews: the role of verbal expression to drive noticing and emphasise; the role of good command of ordinary language; the role of verbalisation as a semiotic mediator between symbolic and visual mathematical expression; and, provisos for (and issues emerging from) the use of ordinary language in mathematical expression.

The interviewees stressed that the mere presence of symbols in a mathematical sentence is not sufficient for driving students’ attention to the key mathematical idea in the sentence. As a helpful and efficient routine that the students currently lack but need to adopt the interviewees propose the assistance on this matter from verbal
adjuncts to the symbolic expression. They offer the definition of convergence as an example of a statement where such assistance can be potent. The premise for their discussion of this example is the following mathematical problem:

Write down a careful proof of the following useful lemma sketched in the lectures. If \( \{b_n\} \) is a positive sequence (for each \( n, b_n > 0 \)) that converges to a number \( s > 0 \), then the sequence is bounded away from 0: there exists a number \( r > 0 \) such that \( b_n > r \) for all \( n \). (Hint on how to start: Since \( s > 0 \), you might take \( \frac{1}{2}s = \epsilon > 0 \) in the definition of convergence.)

Notes on Solutions

Let \( \epsilon = \frac{s}{2} \) in the definition of convergence. Then there is an \( N \) such that \( n > N \Rightarrow |b_n - s| < \frac{s}{2} \Rightarrow b_n \geq \frac{s}{2} \). Then, for any \( n \), \( b_n \geq r = \min\{b_1, \ldots, b_N, \frac{s}{2}\} \) which is the minimum of finitely many positive quantities, hence is positive.

Fig. 1 A Year 1 Calculus question requiring emphasis on the meaning of quantifiers

Typically students responded with omitting a small but significant number of terms:

Fig. 2 Two typical responses to the question with missing emphasis on quantifiers

Discussing what could possibly trigger students’ noticing the need to cover all terms of the sequence, the interviewees highlight the importance of full sentences:

‘It seems that after all the presence of the quantifiers themselves in the text of the question is not emphatic enough to suggest universality or existence to the students. And words, sentences, those creatures ever-absent from students’ writing exist exactly for this purpose: of emphasis, of clarification, of explanation, of unpacking the information within the symbols.’ p151

The more skilled students are in producing such sentences the better the cognitive support they will gain from verbalising their mathematical thought. Hence the interviewees’ discussion of the importance of good command of ordinary language. The following mathematical problem is the premise for their statements on this:
Suppose $A$ in an $n \times n$ matrix which satisfied $A^2 = 0$ (the $n \times n$ zero matrix).

1. Show that $A$ is not invertible.
2. Show that $I_n + A$ has inverse $I_n - A$.
3. Give an example of a non-zero $2 \times 2$ matrix $A$ with $A^2 = 0$.

Fig. 3. A Year 1 Linear Algebra question requiring clarification on implication order

In part (ii) of this question the students need to show that, because the product of $I_n + A$ and $I_n - A$ is $I_n$, $I_n - A$ is the inverse of $I_n + A$. The interviewees notice that one student (Fig. 4) starts her response with a different, and incorrect, statement of her intentions:

Student LD, part (ii)

Fig. 4 A typical response to the question with apparently muddled implication order

This student script was commented upon as follows:

‘I am not very keen on if this is true, then the product of $I-A$ and $I+A$ will be $I$, even though she is doing the absolutely right thing, starting from the product and ending up with $I$. You know why? Because it’s getting so close to appearing as if she is assuming what she is supposed to be proving. What she wants to be saying is really this is true because…. There is a subtlety missing there regarding the converse statement and their Grammar is not up to scratch to help them see the difference.’ p59

Unlike the previous example – in which an emphasis on the need to offer a universal coverage of the sequence’s term could have been provided by a verbal accompaniment to the symbolic expression – here the student’s inaccurate verbal expression does not lead her astray. However the interviewees take the opportunity provided by such inaccuracies in students’ writing in order to stress the strong link between command of ordinary language and ‘good mathematical writing’:

‘It should be made clear to the students that this type of command of the language [for example, people being unable to distinguish between a main and a subordinate clause…] is not irrelevant to good mathematical writing. And that applies all the way through to completion of their studies. I sometimes see final year students and I wonder whether they deserve marks for a response that I could only detect as correct amidst grammatically incorrect statements. When you put things on paper with such ambiguity and inconsistency, such as sentences without verb or subject etc., maybe you should expect a lesser reward too.’ p151

In the above the interviewees state that mathematical writing that is characterised, for example, by grammatical ambiguity and inconsistency deserves lesser rewards. They also wonder whether students’ awareness about the significance of grammatical and syntactical correctness needs to be emphatically raised. In the concluding remarks in this paper I return to this suggestion in order to discuss briefly whether this
appreciation of the role, and employment, of ordinary language in mathematics is perhaps more disproportionately expected from the students than their exposure to, and systematic practising of, this valuable routine reasonably allows for. But before doing so, I first cite the participants’ statements on what they appear to see as the most valuable aspect of verbalisation skills in mathematics: the role of verbalisation as a semiotic mediator between symbolic and visual mathematical expression.

The interviewees cite the definition of convergence as a case illustrating the value of the connection between symbolic, verbal and visual forms of mathematical expression. Students’ first encounters with the hefty symbolisation employed in this definition are some of the first occasions in which students realise that this mode of expression is the discursive norm in mathematical writing and a norm they are expected to accustom to quite quickly. The interviewees stress that verbalisation is a meaningful way to help students face ‘what they see as madness’ at this stage and steer them away from construing the strings of symbols in the definition as little other than ‘formalistic nonsense’. The example of the student in the following is telling:

‘…a student who wrote down a neat response to a convergence question – applying the definition impeccably – and then asked why does this prove the convergence?! What an excellent question! I tried to explain that this is the definition of convergence but students don’t quite understand the relation between this expression and what convergence ought to mean exactly.’ p187

Elaborating on the sources of difficulty with the definition, the following link across visual, verbal and symbolic accounts of the definition was offered:

‘… the difficulty lies with the successive appearance of quantifiers in the definition whereas the primary notion for the students ought to be that no matter what I specify the \( \varepsilon \) region about the \( A \), from a certain point onwards everything fits inside this box. Making this link between this image and the formalisation behind this is utterly important. Otherwise the definition is nothing other than formalistic nonsense.’ p188

The role of verbalisation in investing a symbolic account of the definition with meaning emerges then as crucial:

‘… this is exactly the meaning into words such as eventually and arbitrarily, which I constantly put in my writing in my attempt to convey the idea underlying the formal expression of the definition. Most students however simply ignore them as irrelevant waffle and copy the definitions only in their notes. I even try grouping the various parts of the definition with different colours of chalk!’ p188

But then, if a verbal account offers an opportunity to face difficulty with understanding a complex definition such that of convergence, why, we may wonder, do students typically bypass this opportunity? One explanation is offered below:

‘But using words is risky: I have seen verbal explanations of the definition which are in fact wrong! Like many textbooks are! Which is not embarrassing given the long debates about acceptable verbalisations of the definition. Verbalising, geometrisising it etc. is fine as long as we stay this side of correctness!’ p189
What the above seem to suggest is that resorting to ordinary language is seen as potentially containing some inherent risk, a jeopardy for mathematical precision. Is it therefore conceivable that the students, in their avoidance of, or trepidation towards, verbalisation are simply adopting the risk-averse discourse of the community towards ‘verbalising’ and ‘geometricising’? The following seems to suggest so:

‘… sometimes steering clear of intuitions and pictures etc, yes, working through strings of quantifiers, even though one may not be so sure of what is going on, can be seen as less messy, less risky. I think some students may in fact see it this way and be happy to work this way and just do what they are told. You can view this as the recipe, you can do this, you do this and you do this… You just follow the steps. And in some ways they are safer this way because they will not make mistakes as long as they are technically doing the right steps.’ p189

Wordless discourse seems to exert some allure on mathematicians (see two examples in Fig.5) perhaps because of its capacity to convey meaning with curt elegance. But does this trepidation towards verbalization contrast with – even contradict… – their expectation that students will appreciate and employ verbalisation in mathematics? I conclude with a few thoughts on this potential contradiction.

A CONTRAST BETWEEN EXPECTATIONS AND PRACTICE?

In the above the interviewed mathematicians make a strong case for verbalisation in mathematics: it can drive our noticing of key mathematical ideas and can act as a crucial semiotic mediator between symbolic and visual mathematical expression. They also stress that, for verbal skills to deliver on this potential, good command of ordinary language is important. They observe that students avoid verbalising their mathematical thoughts and they describe this student tendency as missing an opportunity to overcome difficulty with understanding certain complex ideas in mathematics. In discussing certain provisos for what makes verbalisation an acceptable part of mathematical discourse, they cite attention to precision as one such proviso and express their weariness with the potential risk of ambiguity in verbal expression. Their discourse seems to be quite risk-averse and they recognise that students’ avoidance of verbalisation may be underlain by a similar aversion to risk.

What is conspicuously absent in the discussion above is the acknowledgement that, if verbalisation skills are an important part of students’ learning at this stage (with ‘learning’ meant as a ‘change of discourse’ (Sfard, ibid)), then explicit and systematic pedagogical practice has a role to play in facilitating this discursive shift. At least in these interviews the pedagogical strategies employed towards this facilitation appeared mostly implicit in the interviewees’ statements – that they strive for eloquence in their exposition in lectures and that they aim to set a good example, for instance, through their own writing on the board. Given the rather severe absence of mathematical eloquence from most of the student scripts we examined in the course of these interviews, it seems that a more explicit and systematic approach to developing students’ verbalisation skills in mathematics is necessary.
ENDNOTES

1 The interview data sampled here are presented in the format of a re-storied narrative. The narrative approach of re-storying (Clandinin and Connelly, 2000) adopted in this work involves reading the raw transcripts, identifying and highlighting experiences to be told across this raw material and then constructing a new narrative that represents these experiences. So, while fictional, the new narrative is entirely data-grounded. In this sense the interviewee utterances quoted in this paper were constructed entirely out of the raw transcripts of the interviews with the mathematicians. A quotation typifies and condenses the views expressed by a substantial number of participants. (For an example of the re-storying construction process as well as other influences on the data analysis see p27-28 in (Nardi, 2008)). The page numbers attached to each quotation used in this paper correspond to the page in (Nardi, 2008) where the quotation, and its related episode, can be found. The emphasis in bold within the quotations is mine.

ACKNOWLEDGEMENT

The work that I present in this paper draws on studies supported by small grants over recent years by the Nuffield Foundation, the UK’s Learning and Teaching Support Network (Mathematics, Statistics and Operational Research branch) and the UK’s Higher Education Academy. I thank my colleague Paola Iannone for her valuable work with me on these studies.
REFERENCES


