THRESHOLD CONCEPTS: A FRAMEWORK FOR RESEARCH IN UNIVERSITY MATHEMATICS EDUCATION

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A threshold concept is a 'portal' or a 'conceptual gateway' that leads to a previously inaccessible, and initially troublesome, way of thinking about something. A new way of understanding may thus emerge – a transformed view of the subject. The framework of threshold concepts has been used for some years in research of teaching and learning in higher education in several subjects but there are only few articles in mathematics education using this framework. The aim of this paper is to introduce threshold concepts into mathematics education. The result of searching papers in mathematics education using threshold concepts is presented. The need for more research using this framework to create a meeting point for mathematicians and educationalists and to improve students' learning is pointed out.

INTRODUCTION

The aim of this paper is to introduce threshold concepts (Meyer & Land, 2003) as a framework for research in mathematics education. This framework has for some years been used in research of higher education in several subjects but is nearly missing in the mathematics education research. Using the framework of threshold concepts can be very powerful for improving student learning, by providing a language for discussing learning and a meeting point for educational researchers and mathematicians. Threshold concepts have been used successfully to engage subject specialists into pedagogical discussions (Cousin, 2009). My own experience is that the notion of threshold concepts works well also in mathematics (Pettersson, 2008).

Students who are confronted with a mathematical concept sometimes find learning troublesome. There is a large body of research on students' difficulties in shaping their conceptions in accordance with the expectations of the learning environment. For example, the research on students' conceptions in calculus has shown that the learning of several concepts is problematic for the students (for an overview, see e.g. Artigue, Batanero, & Kent, 2007). The concept of limit of functions is one of the problematic instances in the learning of calculus. The concept of limit is a central part of calculus, not only for its own sake, but also because it is used in the definitions of derivative and continuity. Also to get good conceptions of integrals there is a need of a good conception of limit, even though not all definitions explicitly use the concept of limit. 'Limit' is one example of a concept that could be classified as a threshold concept (Meyer & Land, 2003).

The notion of threshold concepts has for some years been used in research in higher education in several subjects. However, in searching for articles in mathematics education using the notion ‘threshold concept’, nearly no findings will appear. In this paper I will first present the theory of threshold concepts and then present some cases using threshold concepts as a framework in economics, engineering and computer science education. I will also present the result of a search for articles using threshold concepts within mathematics education research. Finally I will offer some arguments for using this framework in research on mathematics education. My contribution through this paper is to start a discussion of how focusing threshold concepts can support also research in mathematics education.

WHAT IS A THRESHOLD CONCEPT?

Meyer and Land (2003) introduced the notion *threshold concept*. In students’ efforts to develop their understanding of a subject some concepts may be more crucial than others. A threshold concept can be seen as a ‘portal’ or a ‘conceptual gateway’ that leads to a previously inaccessible, and initially troublesome, way of thinking about something. A new way of understanding may thus emerge – a transformed view of the subject. Threshold concepts are concepts that bind a subject together and the acquisition of such concepts is important to grasping the ways of thinking and practising in the subject (Land, Cousin, Meyer, & Davies, 2005). Threshold concepts are concepts that university teachers typically describe as ‘core concepts’, but it should also be noted that threshold concepts should be regarded first and foremost from a student learning perspective (Meyer & Land, 2003).

A mathematical concept is given by a definition but the definition must be interpreted by a learner. Tall and Vinner (1981) use the notion ‘concept image’ to talk about a student’s interpretation of a concept including everything a student connect to the concept; processes, pictures, examples and also the definition and an interpretation of the definition. Threshold concepts are concepts for which the building of a well connected concept image is troublesome. Meyer and Land (2003) presented the mathematical concepts ‘limit’ and ‘complex number’ as examples of threshold concepts. There are also several examples from other subject areas such as ‘opportunity cost’ in Economics and ‘gravity’ in Physics (Meyer & Land, 2006).

CHARACTERISTICS OF THRESHOLD CONCEPTS

Threshold concepts are characterised as initially troublesome, transformative, integrative, and irreversible (Meyer & Land, 2006). They also tend to serve as boundary markers and may get the students into a liminal space where the students’ understandings are unstable. To identify threshold concepts in a subject area we can search ‘core concepts’ and then use previous research about troublesome learning and students’ conceptions of these concepts. This will give us candidates for threshold concepts for which we through new research can check the characteristics of threshold concepts.

Troublesome knowledge

The learning of a threshold concept is likely to be troublesome for the students. Perkins (1999) defines ‘troublesome knowledge’ as a kind of knowledge which appears counter-intuitive, alien or seemingly incoherent. A way for students to avoid this troublesome knowledge is to remain in a common sense or intuitive understanding of the concept. Pushing students to change their understanding is not unproblematic as it can involve an uncomfortable and emotional repositioning (Cousin, 2009).

Transformative

Understanding a threshold concept will bring about a significant shift in students’ perception of a subject or a part thereof (Meyer & Land, 2003). It is not just a new understanding of the concept; it involves transformation of the understanding of the whole subject area where the concept is located. It includes an ontological as well as a conceptual shift. The shift in perspective may lead to a reconstruction of subjectivity. It is likely to involve an affective component. The transformation may be sudden, but it is mostly stretched over a long period.

Integrative

Threshold concepts are concepts that bind a subject together. Understanding a threshold concept will expose previously hidden relations between concepts in the subject area (Meyer & Land, 2003). Mastery of threshold concepts helps the student to overcome a fragmented view of the subject, things fall into place (Cousin, 2009). Integrating prior understandings is also part of the transformation of understanding of the whole subject area (Davis & Mangan, 2007).

Irreversible

The change in perspective brought about in the course of developing an understanding of a threshold concept is unlikely to be forgotten or will be unlearned only by considerable effort (Meyer & Land, 2003). This does not exclude modification or rejection for a more refined understanding. One problem in this irreversibility is for teachers, who have transformed their understandings, to look back across thresholds to be aware of the kind of understandings the students are likely to have before transformation has occurred (Cousin, 2009).

Boundary markers

Threshold concepts tend to be bounded in that they “serve as boundary markers for the conceptual spaces that constitute disciplinary terrain” (Land, Meyer, & Smith, 2008, p. x). A student entering a new conceptual space by grasping a threshold concept will find that this new conceptual space will be surrounded by other threshold concepts. These threshold concepts form a frontier to new conceptual areas.

Liminal spaces

During the process of mastery of a threshold concept the students may enter a liminal space (Meyer & Land, 2006). The notion is taken from the word *limen*, a Latin word meaning boundary or threshold. This space could be compared with the period of adolescence; not yet being an adult, not quite a child. In this unstable state the learner may oscillate between old and new understandings just as adolescents move between acting as a child and as an adult. These liminal spaces are spaces of transformations. Most of the students in this state will oscillate between grasping the concept and then loosing that grasp. A learner engaged with the project of mastering a threshold concept would enter the liminal space and hopefully the learner will come out from this space with a transformed understanding of the concept. However, the liminal space can also become a 'stuck place' for the student. Some students will then fake understanding through the practice of mimicry, learning by rote or learning how to solve typical problems (Cousin, 2009).

THRESHOLD CONCEPTS IN SEVERAL DISCIPLINES

Over the past few years, research using the framework of threshold concepts has been carried out in different subjects, such as economics, engineering and computer science (Land, Meyer, & Smith, 2008). Starting with the ETL-project, Enhancing Teaching-Learning Environments in Undergraduate Courses, in UK where several institutions took part, threshold concepts is "now moving from a position of being a leading edge new perspective to one which is catching the interests of academics educational researchers in a growing number of countries" (Meyer & Land, 2006, p.xii).

In a project aimed at empirically identifying threshold concepts in computer science Zander, Boustedt, Eckerdahl, McCartney, Moström, Ratcliffe, and Sanders (2008) used interviews and questionnaires to obtain data from teachers and students. 'Object-oriented programming' and 'memory/pointers' emerged as candidates for threshold concepts. The project also investigated how students understand these concepts with the aim to improve teaching and learning.

Shanahan and Meyer (2006) studied a threshold concept in Economics; 'opportunity cost'. The results points out that there are important implications for the manner in which students are introduced to threshold concepts. When learning threshold concepts 'first impressions matters'. Efforts to make the concept easier by simplifying students' first impressions may set students on a path of ritualised knowledge that creates barriers to a deeper understanding and prevents students from crossing the threshold.

In electrical engineering Carstensen and Bernhard (2008) considered the troublesome concepts ‘frequency response’. They propose that certain concepts can function as a ‘key’ that opens up the portal of understanding. Teaching such ‘key’ concepts do not just open up for understanding of that concept, but also the learning of other concepts related to it. Using the tool ‘Bode Plots’ to illustrate frequency response was found to function as such a ‘key’. The results from focusing the teaching on these ‘key’ concepts opening up for understanding the threshold concepts showed an improvement in the scores achieved by students on a test at the end of the course.

Just a few examples are given within this paper; many more findings have been published (e.g. Land, Meyer & Smith, 2008). Recently a new book on threshold concepts was published (Meyer, Land, & Baillie, 2010). However, there is a lot more to do, especially in relation to mathematics education.

THRESHOLD CONCEPTS IN MATHEMATICS EDUCATION

The notion of threshold concept is established in the research of higher education in general but searching for articles in mathematics education using the notion threshold concepts gives just a very few findings. Using the database ERIC and the combination ‘mathematics’ AND ‘threshold concept’ searching in all text produced five hits when searching in the period of 2003-2010 (access 2010-09-06). Two of these five articles are about ‘threshold values’ of variables used in physics and resources on the Internet. The remaining three articles are published in Educational studies in mathematics (Williams, 2009), European journal of engineering education (Masouros & Alpay, 2010) and Higher education (Scheja & Pettersson, 2010).

Using the database Academic Search Premier and the combination ‘mathematics’ AND ‘threshold concept’ searching Boolean/Phrase in all text produced 32 hits of peer reviewed articles when searching in the period of 2003-2010 (access 2010-09-06). However, scanning the list of these articles only seven are in the area of mathematics education; that is reporting research about teaching and learning in mathematics courses. These seven articles include the three articles fund in ERIC. The other four are published in Educational studies in mathematics (Bramby, Harries, Higgins, & Suggate, 2009), European journal of engineering education (McCartney, Boustedt, Eckerdahl, Moström, Sandres, Thomas, & Zander, 2009; Booth, 2008) and International journal of mathematical education in science and technology (Pettersson & Scheja, 2008).

Threshold concepts are in focus in two of the articles. Scheja and Pettersson (2010) discuss the transformative aspect of threshold concepts suggesting that the transformation involves a transformation of the students’ conceptions as these develop through shifting contextualisations of the concepts. McCartney and colleagues (McCartney et. al., 2009) also focuses threshold concepts, but mostly related to software engineering. The article summarizes findings concerning how computer science students experience the liminal space and discuss how this might affect teaching.

In the other articles threshold concepts are more or less just mentioned. Masouros and Alpay (2010) focuses on the design of a computer-based mathematics resource. When discussing the mathematics content of the resource they emphasize that special effort should be given to troublesome topics that lead to a transformation in understanding, mentioning threshold concepts and referring to Meyer and Land (2003).

Williams (2009) puts forward the notion ‘threshold moment’ where seeing and grasping at the nexus of two or more activities often seem to be critical to breakthroughs in learning. In a footnote Williams makes a note on literature on ‘threshold concepts’ and also points out that not very much has been published about threshold concepts in mathematics.

Barmby, Harries, Higgins, and Suggate (2009) discuss children’s understanding and reasoning in multiplication. When the authors talk about key representations for a concept they relate this to the notion of ‘key development understandings’ defined by Simon (2006). They make a quotation from Simon and in this quote parallels with the notion of ‘threshold concepts’ are mentioned. Barmby and colleagues do not take this any further.

Booth (2008) addresses the issue of teaching and learning engineering mathematics based on a form of understanding that goes beyond facts, theorems and algorithms. She points out that the mathematicians as mathematics teachers in the engineering education mostly are interested in the learning objects, in what to learn, also mentioning that educational researchers have much knowledge about the ways students might be comprehending the learning objects, “in particular when it comes to ‘threshold concepts and troublesome knowledge’ ” (p. 383) referring to Meyer and Land (2003).

Pettersson and Scheja (2008) explore the nature of students’ understanding of the concepts limit and integral. As a reason to study these concepts it is argued that these concepts are threshold concepts giving references to Meyer and Land (2005, 2006) and to previous research about students’ conceptions in calculus.

Looking at these articles found by searching databases it could be seen that the notion threshold concepts until now is just used in really few research articles in mathematics education. In the next paragraph I point out that this framework is very useful also in mathematics education.

TO BE USED IN MATHEMATICS EDUCATION

Using the framework of threshold concepts is potentially even more successful in mathematics than in other subjects. In mathematics the concepts are an important part of the curriculum and the concepts are explicitly defined in a way that is uncommon in other subjects such as economics and engineering. The concepts and the relation between them build the core of mathematics (Davis & Hersh, 1990).

In my own research (Pettersson, 2008; Pettersson & Scheja, 2008; Scheja & Pettersson, 2010) I have studied students' conceptions of function, limit, derivative and integral. Looking through previous research about students' learning of calculus (e.g. Artigue et. al., 2007) these concepts are candidates as threshold concepts. The characteristics of threshold concepts, as integrative and transformative in the learning process, points out the importance of research about conceptions of these concepts. Pettersson and Scheja (2008) points out the students' algorithmic way of interpreting the concepts of limit and integral. Looking at these concepts as threshold concepts and focusing on the transformative aspect of the threshold showed that the students started to pass the threshold (Scheja & Pettersson, 2010). Challenging questions in the interviews showed to be important to start this transformation. Threshold concepts can in this way be used to focus on powerful transformation points in the students' learning; they are 'jewels in the curriculum' (Land, Cousin, Meyer, & Davies, 2006, p. 198). They give the students opportunities to develop fundamental conceptual understanding. Using threshold concepts in teaching a subject offers an opportunity to focus on the points that are really useful in mastering the subject (cf. Carsetensen & Bernhard, 2008). Focusing the most important concepts of the subject will also be a way of avoiding an overstuffed curriculum (Cousin, 2009).

Threshold concepts are characterized as potentially troublesome. The students are likely to encounter troublesome knowledge and experience conceptual difficulty. These concepts are potentially 'stuck points'. The students get into the liminal space with uncertainty and oscillation in understanding. This unsecure phase of learning may be necessary and unavoidable but we do not want the students to stay for a long time in the liminal space. Scheja and Pettersson (2010) argue that the students' shifting from an algorithmic contextualisation to a contextualisation inviting reflection on conceptual dimensions of limit and integral is a way to move on from the liminal space. Threshold concepts are rarely mastered at a specific point of time, an 'aha' moment, mastery might take years to complete (McCartney et. al., 2009). However, teachers listening to the students' uncertainties give possibilities for helping the students through the liminal space. Knowledge about these potentially stuck places is important in the teaching. Shanahan and Meyer (2006) pointed out the importance of the first impression; to simplify the threshold concepts do not help the students.

One of the big advantages of research on threshold concepts is that it animates interests and discussions among academics. This kind of educational research gives centrality to the subject specialists and gives a platform for partnership between educational researchers and subject specialists. Such a focus on threshold concepts is also a good way to involve both subject specialists and educational researchers in a discussion on student learning, curriculum design and teaching the subject. Getting academics together to discuss and identify threshold concepts in their subject area has proved to be very fruitful (Cousin, 2009). My own experience of discussing teaching and learning mathematics at university with mathematicians is that threshold concepts appear to fit very well in mathematics. Presenting the idea of threshold concepts to mathematicians always bring up discussions about what threshold concepts there can be in calculus, in linear algebra and other subject areas. These discussions usually spark ideas about students' conceptions and how to improve student learning.

The lack of research in mathematics education using the powerful framework of threshold concepts indicates a need of more research in this area. This kind of research has the possibility to improve the teaching and learning of mathematics. The framework will be powerful in research of university mathematics education but it can also be used in research on mathematics teaching and learning at other levels such as primary and secondary school.

CONCLUSIONS

The framework of threshold concepts has successfully been used in educational research into several subjects. However, there are still few publications related to mathematics education. Using this framework in mathematics education will contribute to mathematics education in several ways. There are several concepts in the subject of mathematics that are troublesome for the learners. Knowledge about concepts which are transformative and integrative will help us to improve teaching and learning. To focus on threshold concepts is a way to avoid an overstuffed curriculum. Research has also pointed out that the students scored higher when the teaching was focusing threshold concepts (Carstensen & Bernhard, 2008). The notion of threshold concepts also creates a meeting point for educationalists and mathematicians. The notion provides a language for discussions about how to improve teaching and learning. There is a growing body of results about threshold concepts in several subjects, but there is a lot more to do in relation to mathematics.

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