THE USE OF MATHEMATICS SOFTWARE IN UNIVERSITY MATHEMATICS TEACHING

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In this paper we report on a teaching experiment regarding the definite integral concept in university mathematics teaching. The experiment was carried out at a Swedish university by using the Free Dynamic Mathematics Software GeoGebra. In our theoretical framework we apply Variation Theory, originating in the phenomenographic research tradition. The data of this study consist of the lecture plan and the engineering students' answers to pre and post tests. In the analysis of the data we applied statistical methods. The experiment revealed that by using the GeoGebra it is possible to create learning opportunities of the definite integral concept that support the students' learning.

INTRODUCTION

There is a constantly increasing number of software packages that can be used as a powerful tools in mathematics teaching. Recent research shows that computer programs such as Maple, Mathematica, Derive, Geometer's Sketchpad, GeoGebra, when used in a classroom support creative discoveries and mathematical generalizations (Lavicza, 2006).

It has been shown that students who use technology in their learning had positive gains in learning outcomes over students who learned without technology (Camacho, Depool & Sant-Trigo 2010; Machín & Rivero, 2003; Touval, 1997). Regardless of evidences of several benefits of using technology, the process of applying technology for mathematics education is slow and complex (Cuban, Kirkpatrick & Peck, 2001).

Several studies have highlighted difficulties that students encounter with the integral concept. In an early study carried out by Orton (1980) it was observed that students had difficulty with the integral $\int_a^b f(x)dx$ when *f(x)* is negative or *b* is less than *a*. In another study by Orton (1983) it was noticed that some students found it difficult to solve problems related to the understanding of integration as a limit of sums.

Earlier research (Attorps, Björk, Radic & Tossavainen, in press; Blum, 2000; González-Martín & Camacho, 2003) has even pointed out that students have an intention to identify the definite integral as an area. Further, Rasslan and Tall (2002) verified that a majority of the students cannot write meaningfully about the definition of the definite integral. In similar way, studies concerning learning of calculus concepts (Attorps, 2006; Rösken & Rolka, 2007; Viirman, Attorps & Tossavainen, in press) have shown that definitions play a marginal role in students' learning whereby intuition inherent in concept images dominates the concept learning.

Transformation from procedural to conceptual understanding of the concept of integral requires gradual reconstructions of students' perceptions. Research has however documented the limitations of standard teaching methods, showing that students become reasonably successful on standard tasks and procedures but have difficulties in developing a solid and conceptual understanding of the topics itself (Artigue, 2001).

The Variation Theory

The variation theory is a theory of learning, which is based on the phenomenographic research tradition and described by Marton and Booth (1997). The main idea in the phenomenography is to identify and describe qualitatively different ways in which people experience certain phenomena in the world, especially in an educational context. There are two main principles in variation theory. The first one is that learning always has an object, in our case the definite integral concept. The second one is that the object of learning is experienced and conceptualized by learners in different ways.

The object of learning can be seen from teacher's, student's and researcher's perspectives. The *intended* object of learning is the object of learning as seen from the teacher's perspective. It includes what the teacher says and wants the students to learn during the lecture. The students experience this intended object of learning in their own way and what they really learned - the outcomes of learning - is called the *lived* object of learning. Hereby, it is easy to understand that students' learning does not always correspond to what the teacher's intention was with the lecture. The *enacted* object of learning as seen from the researcher's perspective defines what is possible to learn during the lecture, to what extent, and in what forms the necessary conditions of specific object learning appear in a classroom setting. The enacted object of learning describes the space of learning, which students and teacher have created together. In this space it is possible for students from their previous learning experiences to discern the critical aspects of the object of learning. (Marton, Runesson & Tsui, 2004).

In variation theory necessary conditions for learning are the experience of *discernment, simultaneity* and *variation*. Variation is a main concern in this theory and a primary factor in supporting student learning. In order to understand what variations to use in the classroom to promote student learning, it is first necessary to understand the varying ways students experience something. We can use this information to identify ways to encourage students to discern other aspects of the learning object, the aspects they have previously not discerned.

Every concept, situation or phenomena have particular aspects and if an aspect is varied and another remained invariant, the varied aspect will be discerned. The understanding of the object of learning in a certain way requires the simultaneous discernment of critical aspects of the object of learning (Marton & Morris, 2002; Marton, Runesson & Tsui, 2004). The theoretical elements - *discernment, simultaneity* and *variation* related to learning and supposed to be critical for learning to happen - can be also used as an analytical tool for analyzing teaching (ibid). As a result, learning and teaching is brought closer together.

The purpose of the study

The aim of our study is to design teaching sequences for definite integrals, using the GeoGebra software, that can support the students' learning in university mathematics. To that end, we seek an answer to the following question: Is it possible to apply GeoGebra as a pedagogical tool within the variation theory, in order to vary the critical aspects of the concept of the definite integral during a lecture on an introductory calculus course?

METHOD AND DESIGN OF THE STUDY

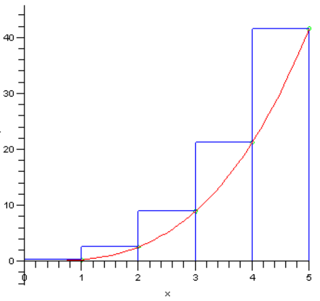
The study took place during a lecture in mathematics at a Swedish university. A total of 17 Chinese engineering students were involved in our study. The data were gathered by doing pre and post tests concerning the definite integral concept. In the analysis of the pre and post test results statistical methods were applied.

The questionnaire

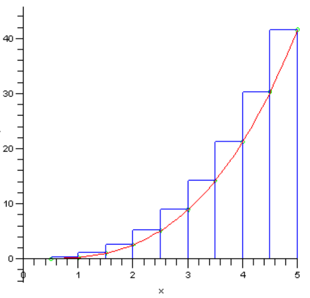
The questionnaire contained 6 questions, including not only 'typical' questions (Questions 2, 3, 5 and 6) but also intuitive questions (Questions 1 and 4). Maximal point in each question was three. Students had 25 minutes to do this test. In both pre and post test the same questionnaire was used. It was not allowed to use technical facilities.

Question 1: If you want to calculate the area between the curve and x-axis when x=0 and x=5 (see the graphs below) you can get an approximate value of this area by calculating the areas of the column and by adding them.

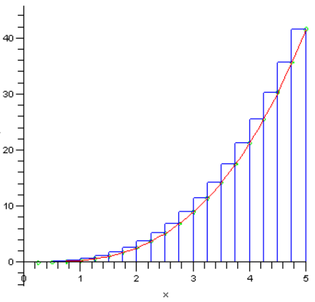
a) Which of the following graphs should you choose in order to make the error as small as possible?



Graph 1



Graph 2



Graph 3

b) Can you answer why?

The aim of the first question was to test the students' intuitive conception about the definite integral concept as a limiting process. In this case it is about the upper Riemann sum.

Question 2. What is this $\int_a^b f(x)dx$ (the definite integral to the function $f(x)$ in the interval [a, b]) according to your opinion?

The second question was to test how the students grasp the conception of the definite integral.

Question 3. There are some approximate values of x and $F(x)$ below:

x	1	2	3	4	5
$F(x)$	-1	-0.61	0.30	1.55	3.05

You know that $F'(x) = \ln x.$ Approximate the value of $\int_3^5 \ln x dx.$

The purpose of the third question was to test how the students can apply the Fundamental Theorem of Calculus [1].

Question 4. The following is given: $\int_{-1}^{5} f(x)dx = 2 \textit{ and } \int_{-1}^{7} f(x)dx = -1.$

Evaluate $\int_5^7 f(x)dx.$

The intent of the forth question was to test the students' knowledge of the properties of the definite integral concept.

Question 5. Can you find any error in the following reasoning?

$$\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{1} x^{-2}dx = \left[\frac{x^{-1}}{-1}\right]_{-1}^{1} = \frac{-1}{1} - \frac{-1}{-1} = -2$$

The aim of the fifth question was to test if the students grasped when it is possible to apply the Fundamental Theorem of Calculus.

Question 6. Find the area of the region, which is limited by the functions $f(x) = 0,5x^2$ and $g(x) = x^3$. Give an exact answer.

The idea of the last question was to test how the students comprehended an ordinary high school example of the definite integral. We also wanted to find out how the students could master calculation with fractions.

RESULTS

Before designing our lecture we analysed carefully the pre test results in our study. The aim of the pre test was to identify if the students have some initial conceptions concerning the definite integral concept. Although the concept in China is normally introduced first at university level (Wang, 2008) we could find that most of the students showed quite a good intuitive conception of the concept as a limiting process (Question 1) and of the properties of the definite integral concept (Question 4). By reading earlier research we could find that many students often have only an area conception of the definite integral (see e.g. González-Martín & Camacho, 2003). Furthermore they cannot meaningfully define the definite integral (Rasslan & Tall, 2002). Having this information about the students' conceptions we designed our lecture. In order to create different teaching sequences that could encourage students to discern varying aspects of the object of learning we used GeoGebra Software.

In the first application of GeoGebra (Figure 1) we introduced numerical approximation of the area (Lower and Upper Riemann sums) as well as the definition of the definite integral with inherent infinite processes.

Figure 1 visualizes the concept of the Riemann integral using lower and upper sums. Two points a and b are shown that can be moved along the x-axis in order to modify the investigated interval. The upper and lower values together with their differences are displayed as dynamic text which automatically adapts to modifications. In this case we keep f(x) and the interval invariant and vary the number of subintervals. By increasing the number of subintervals we shorten their length. Our intention was to show that increasing of the number of subintervals decreases the difference between the Lower and Upper Riemann sums, which shows that the Lower and Upper Riemann sums eventually coincide with the value of the integral.

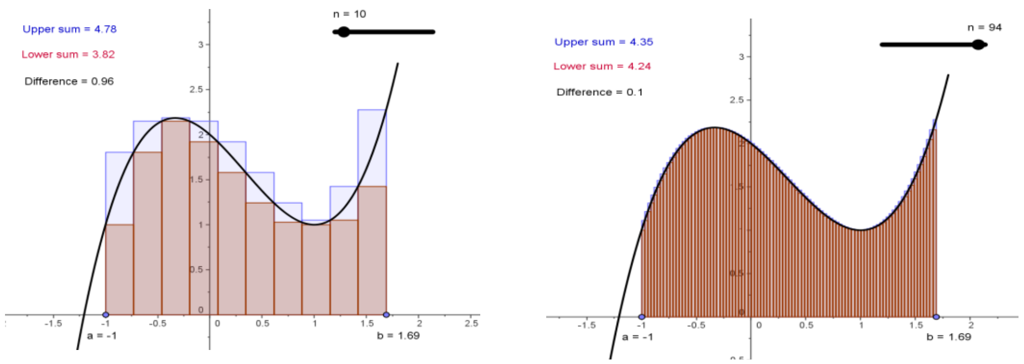


Figure 1: Lower and Upper Riemann sums and inherent infinite processes

The second examples (Figure 2 and 3) should help the students to get a wider conception of the definite integral concept.

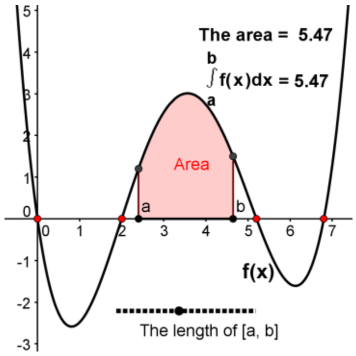


Figure 2: The value of the definite integral is identical to the area between function and x-axis in the interval [a, b]

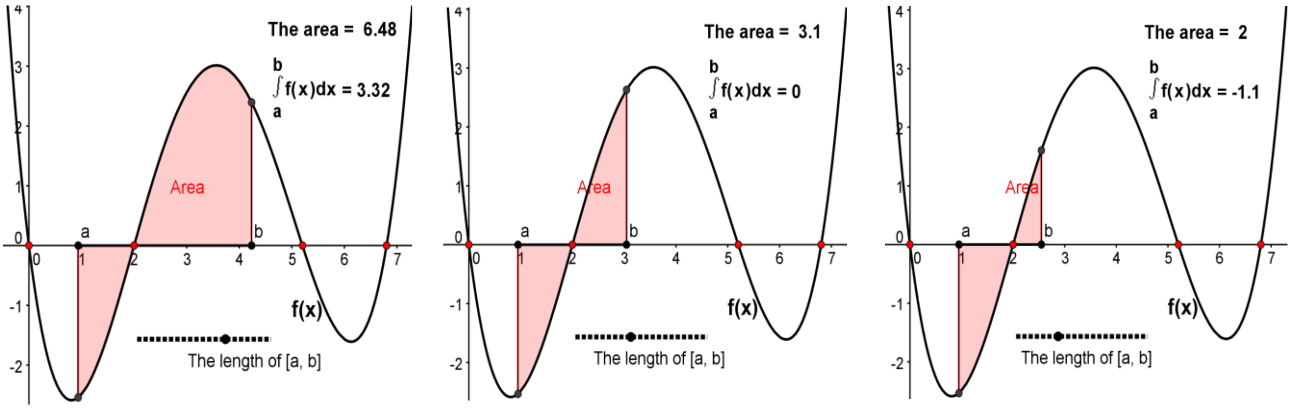


Figure 3: The definite integral as a real number, which can be positive, zero or negative

Figures 2 and 3 visualize the Riemann integral related to the area between the function f(x) and the x-axis. Two points a and b are shown that can be moved along the x-axis in order to modify the investigated interval. The area and the integral values are displayed as dynamic text which automatically adapts to modifications. This time we keep only f(x) invariant and vary both the length of the interval and the upper and lower limit points. Our aim with this teaching sequence was to show that the value of the area between the function and the x-axes and the integral-value not always coincide. While the area is always a non-negative (but not necessarily constant) real number, the integral-value can be any real number.

Our ambition with the third presentation (Figure 4) was to help the students to discern situations when it is possible to apply the Fundamental Theorem of Calculus and when it is not.

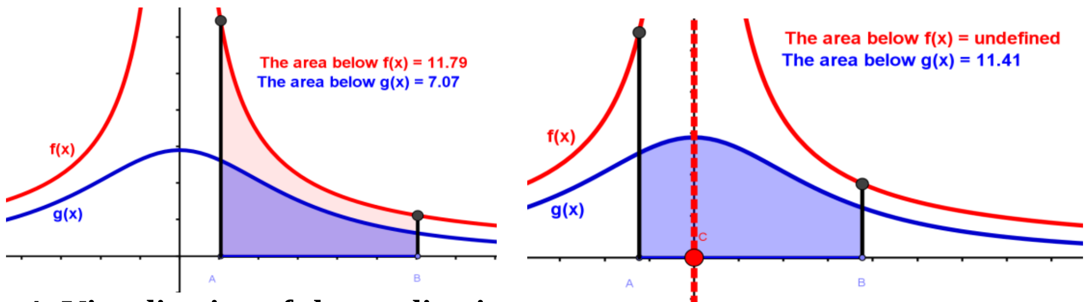


Figure 4: Visualization of the application of Fundamental Theorem of Calculus

In figure 4, on the left hand side, it is possible to apply the Fundamental Theorem of Calculus, because both of the functions *f(x)* and *g(x)* are defined and continuous in the closed and bounded interval from A to B. In figure 4, to the right, it is not possible because f(x) is not defined and by that not continuous in the interval. By moving the point A along the x-axis we can vary the position of the investigated interval. In this teaching sequence we keep the length of the interval and the functions f(x) and g(x) invariant.

The analysis of the data from the pre and post tests was done with a statistic program, the Minitab package. Using the significance level of 95% and a paired-samples one-tailed *t*-test we compared the means of the test results for each problem. The group consisted of 17 participating Chinese students.

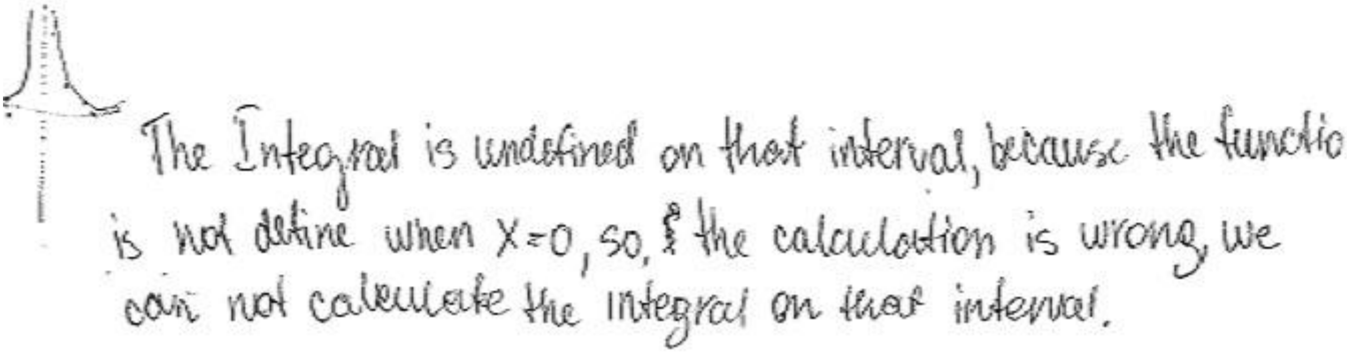
The pre and post test results indicate that there are statistically significant improvements in all the questions after the GeoGebra designed teaching practice.

Table 1: Pre and post test results for the group of Chinese students

Question number	Pre test mean	Post test mean	p	Maximum scores
1	1.588	2.059	0.014*	3
2	0.059	0.647	0.000*	3
3	0.176	2.471	0.000*	3
4	0.529	2.647	0.000*	3
5	0.000	0.941	0.007*	3
6	0.118	2.765	0.000*	3

* $p < 0.05$

The students' scores in question 1 show that the students' intuitive understanding of the definite integral concept was quite good already at the beginning. One of the students explained in the post test the question 1 on the following way: "The difference between the area of columns and the curve is smaller as the columns become smaller and smaller, more and more". The scores in questions 2 and 5 remained low in both pre and post tests. The most typical explanation to question 2 was: "Points a and b are the intersection points of the two functions and we can calculate the area". So we could notice that most of the students still grasped the integral concept as an area. Pre and post test results in question 5 showed that many of the students failed to give an adequate response in question 5; most of them could not find any errors at all. One of the students who succeeded to motivate his answer in post test explained:



We also noticed that nearly all the Chinese students in the post test were successful in question 6, showing good ability to calculate with fractions.

DISCUSSION

The integrating of mathematical software in teaching and learning at the university level is important due to its ability to give quick feedback and help students to visualize and discern simultaneously varying aspects of the object of learning (Marton & Morris, 2002). One of the aspects is the understanding of the definite integral in a wider context as a real number and not only an area. We could observe that the use of the GeoGebra software during the lecture increased the students' possibilities to experience the intended object of learning, that is the concept of the definite integral as a real number. In our post test results we could notice that most of the students still grasped the integral concept as an area (cf. González-Martín & Camacho, 2003). Hereby, the students' learning didn't correspond to what our intention was with the lecture. In our opinion the definite integral concept is too tightly connected with the area conception both in textbooks used on upper secondary and even on university level. Since in traditional classroom settings typical examples from the textbooks are primarily used to introduce a new concept, it is not so surprising that the students often have narrow conceptions of the mathematical concepts. The acquired experiences from the concept learning seem to be too solid and perhaps prevent adequate learning of the intended mathematical theory.

The understanding of the definite integral concept and the Fundamental Theorem of Calculus unavoidably requires that a student must at the same time focus on quite many separate elements of knowledge. Many of them are given in a symbolic or implicit form, that assume that learners can distinguish several aspects of the concept simultaneously. Pre and post test results showed that many of the students failed to give an adequate response in question 5. We think that the students should be trained to use definitions as an ultimate criterion in teaching and learning of mathematics. We see a clear potential in using GeoGebra within the variation theory but we mean that it covers only one of the several representations, which could be used in mathematics teaching.

Further studies need to be undertaken to identify which other factors than the integration of technology in teaching and learning of mathematics that can benefit both educators and students. Therefore, it would be interesting to design a study by choosing two groups of students; in one group lectures on Calculus will be conducted without any software package and in the other group the teaching block of Calculus will be created by using GeoGebra.

NOTES

1. Suppose that

   a) a function f(x) is defined and continuous on the interval [a, b]

   b) F(x) is an antiderivative of f(x) on the interval, i.e. F'(x) = f(x) for all x in [a, b]. Then

$$\int_a^b f(x)dx = F(b) - F(a)$$ (Adams, 2006)

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