MATHEMATICAL OBJECTS THROUGH THE LENS OF THREE DIFFERENT THEORETICAL PERSPECTIVES

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In this paper we establish a link between the onto-semiotic approach (OSA) to mathematics cognition and instruction, APOS theory and the cognitive science of mathematics (CSM) as regards their use of the concept 'mathematical object'. It is argued that the notion of object used in the OSA does not contradict that employed by APOS theory or the CSM, since what the latter two theories do is highlight partial aspects of the complex process through which, according to the OSA, mathematical objects emerge out of mathematical practices.

Key words: mathematical object, onto-semiotic approach, APOS theory, cognitive science of mathematics.

INTRODUCTION

One characteristic of the research community in mathematics education is its diversity of different theoretical perspectives, and hence there is a need for strategies that connect theories. Each theoretical perspective tends to privilege certain dimensions of reality over others. Thus, it is not always easy to identify links between research questions, descriptions, methodologies and conclusions that are elaborated within different paradigms. In this paper we establish a link between the OSA, APOS theory and the CSM as regards their use of the concept 'mathematical object'.

We begin by arguing that the notion of 'object' used in APOS theory is the result of a cognitive process referred to as 'encapsulation', which, in our opinion, is no different from the process of reification. Hence, this is a cognitive view of mathematical objects. We then argue that in the CSM the notion of object emerges through the metaphorical projection of a series of image schemas, and especially that it is the result of the object metaphor. Finally, it is argued that the notion of object in the OSA is broader than that used by the other two theories, although it does not contradict them. This is because what both APOS theory and the CSM do is highlight partial aspects of the complex process through which, according to the OSA, mathematical objects emerge out of mathematical practices.

THREE DIFFERENT THEORETICAL PERSPECTIVES

The OSA

Figure 1 (Font & Contreras, 2008, p. 35) shows some of the theoretical notions contained in the onto-semiotic approach to mathematics cognition and instruction (Godino, Batanero & Font, 2007). Here mathematical activity plays a central role and is modeled in terms of systems of operative and discursive practices. From these practices the different types of related primary objects (language, arguments, concepts, propositions, procedures and problems) emerge, building epistemic or cognitive (depending on whether the adopted point of view is institutional or personal) configurations among one another (see hexagon in Figure 1).

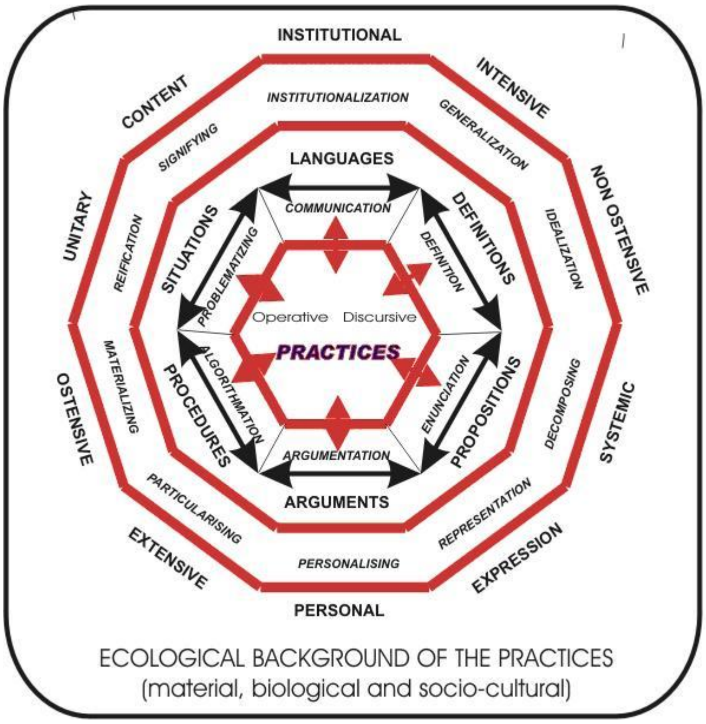

Figure 1. An onto-semiotic representation of mathematical knowledge

Problem-situations promote and contextualize the activity, languages (symbols, notations, graphs) represent the other entities and serve as tools for action, and arguments justify the procedures and propositions that relate the concepts. Lastly, the objects that appear in mathematical practices and those which emerge from these practices depend on the "language game" (Wittgenstein, 1953) in which they participate, and might be considered from the five facets of dual dimensions (decagon in Figure 1): personal/institutional, unitary/systemic, expression/content, ostensive/non-ostensive and extensive/intensive. Both the dualities and objects can be analyzed from a process-product perspective, a kind of analysis that leads us to the processes shown in Figure 1. Instead of giving a general definition of process the OSA opts to select a list of processes that are considered important in mathematical activity (those of Figure 1), without claiming that this list includes all the processes implicit in mathematical activity; this is because, among other reasons, some of the most important of them (for example, the solving of problems or modeling) are more than just processes and should be considered as hyper- or mega-processes.

The APOS theory

APOS (Asiala et al., 1996; Dubinsky & McDonald, 2001) is an acronym that stands for the types of mental structures (Action, Process, Object, and Schema) which students build in their attempts to understand mathematical concepts. A learner, to whom we usually refer as a student, uses certain mental mechanisms called interiorization, coordination, and encapsulation to construct these structures. According to APOS theory, the formation of a mathematical concept involves applying a transformation to existing objects to obtain new objects. An action is any transformation of objects according to an explicit algorithm in order to obtain other objects, and is seen as being at least somewhat externally driven.

As an action is repeated and the individual reflects upon it, it may be interiorized into a mental process. An important characteristic of a process is that the individual is able to describe, or reflect upon, the steps of the transformation wholly in her/his mind without actually performing those steps. Additionally, once a mental process exists, it is possible for an individual to think of it in reverse and possibly construct a new process (a reversal of the original process).

When an individual becomes aware of the process as a totality and is able to transform it by some action, we say that the process has been encapsulated as an object. When necessary, an individual may de-encapsulate an object back to its underlying process. In other situations, the individual may think of the transformation in terms of actions.

A schema for a certain mathematical concept is an individual's collection of actions, processes, objects and other schemas linked consciously or unconsciously in a coherent framework in the individual's mind.

The research method or investigative approach of this framework consists of three-step cycles. The first step is a theoretical analysis of the actions, processes, objects, and schemas that a learner may construct in order to learn a given/specific mathematical concept. The resulting description is called a genetic decomposition of the concept. This is used to design and implement the second step, the instructional treatment. The third step is the collection and analysis of both quantitative and qualitative data.

Cognitive Science of Mathematics

In this paper we are interested particularly in Lakoff and Nunez's account. This is a very particular and, may be, controversial interpretation of "cognitive science of mathematics". Lakoff and Núñez (2000) state that the mathematical structures people build have to be looked for in daily cognitive processes, such as image schemas and metaphorical thinking. These processes allow us to explain how the construction of mathematical objects is supported by the way in which our body interacts with the objects of everyday life. To achieve abstract thinking we need to use basic schemas derived from the immediate experience of our bodies. We use these basic schemas, called image schemas, to make sense of our experiences in abstract domains through metaphorical mappings. Lakoff and Núñez (2000) claim that metaphors create a conceptual relationship between the source domain and the target domain. They distinguish between two types of conceptual metaphors in relation to mathematics: a) grounding metaphors that relate a source domain outside of mathematics with a target domain inside mathematics; and b) linking metaphors that have their source and target domains within mathematics.

THE LINK BETWEEN THESE FRAMEWORKS

The emergence of objects in the OSA

The OSA considers that the process through which mathematical objects emerge out of mathematical practices is a complex one and that at least two levels need to be distinguished. The first level corresponds to the emergence of representations, definitions, propositions, procedures, problems and arguments (primary objects). As regards the nature of these objects, the OSA, in line with the philosophy of Wittgenstein, considers that the type of existence which can be ascribed to the concepts/definitions, the propositions and the procedures is that of conventional rules. From this point of view, mathematical statements are rules (of a grammatical kind) for using certain types of signs, because in fact they are used as rules. They do not describe properties of mathematical objects that exist independently of the people who wish to know about them or of the language used to know them, even though it may appear that this is the case.

Although the OSA adopts a conventionalist point of view on the nature of mathematical objects, it is acknowledged that a descriptive/realist view of mathematics is implicitly suggested in teaching processes. In order to explain how this vision is generated it is necessary to consider a second level in the emergence of mathematical objects; an example might be the object ‘function’, which is considered as an object that is represented by different representations that may have several equivalent definitions, which have properties, etc.

In order to explain how primary objects emerge, the metaphor of ‘climbing stairs’ proves highly useful. When we climb stairs we have to stand on one foot as we move, but that foot then moves progressively to a higher stair. Mathematical practice can be considered as ‘climbing stairs’. The stair on which we stand in order to carry out the practice is an already-known configuration of primary objects, whereas the higher stair which we then reach, as a consequence of the practice carried out, is a new configuration of objects, one (or more) of which was previously unknown. The new primary objects appear as a result of this mathematical practice and become institutional primary objects due, among other processes considered in Figure 1 (including reification and idealization), to processes of institutionalization that form part of the teaching-learning process being studied.

The second level of emergence is the result of several factors, the main ones being as follows: 1) mathematical discourse, explicitly or otherwise, gives students the message that mathematics is a ‘certain’, ‘true’ or ‘objective’ science; 2) the predictive success of the sciences that make use of mathematics is used, explicitly or otherwise, to argue in favor of the existence of mathematical objects; 3) the simplicity that derives from postulating the existence of mathematical objects. Their postulation is justified on the basis of the practical benefits, especially as regards simplifying the mathematical theory which is being studied. Indeed, it is highly convenient to consider that there exists a mathematical object that is represented by different representations, which can be defined by various equivalent definitions, or which has properties, etc. 4) The object metaphor is always present in teachers' discourse because here the mathematical entities are presented as "objects with properties". It is common in mathematics discourse to use certain metaphorical expressions which suggest that mathematical objects are not constructed but, rather, are discovered as pre-existing objects; for example, words such as 'describe' or 'find', etc. 5) As discussed in Font, Godino, Planas and Acevedo (2010) it is possible in mathematics discourse (a) to talk about ostensive objects representing non-ostensive objects that do not exist (for example, we can say that f'(a) does not exist because the graph of f(x) has a pointed form in x = a), and (b) to differentiate the mathematical object from one of its representations. Both aspects lead students to interpret mathematical objects as being different from their ostensive representations.

These five factors generate, implicitly or explicitly, a descriptive/realist view of mathematics which considers (1) that mathematical propositions describe properties of mathematical objects, and (2) that these objects have a certain kind of existence that is independent of the people who encounter them and the language through which they are known. This view is hard to avoid since the reasons why it is adopted are always operating, albeit subtly. More than a consciously assumed philosophical position we are dealing here with an implicit way of understanding mathematical objects.

Objects in APOS theory

In APOS theory (Asiala et al., 1996) encapsulation and de-encapsulation play an important role. APOS theory begins with actions and moves through processes to objects. These are then integrated into schemas which can themselves become objects. The ideas arise from attempts to extend the work of Jean Piaget on reflective abstraction in children's learning to the level of collegiate mathematics.

In the paper titled "Reification as the Birth of Metaphor", Sfard (1994) reports on the interviews she conducted with three renowned mathematicians. In these interviews the three mathematicians talk about the mathematical concepts that they study as if they were concrete in some way. The term that Sfard uses for this cognitive phenomenon is reification, which is similar to Dubinsky's construct of encapsulation.

Reification is a term used in philosophy that means, etymologically, "to treat something like a thing". In the processes of mathematical reification, abstract notions are conceived like objects. To reify (or encapsulate) is to regard or treat an abstraction as if it had a concrete or material existence. One example of reification is when we assume — or state linguistically — that there is an object with various properties or various representations.

The link between the OSA and APOS theory

In accordance with the onto-semiotic approach (see section 3.1) we consider that reification (encapsulation) is very important in terms of explaining the emergence of mathematical objects, but that it is insufficient to describe adequately this emergence and the nature of mathematical objects. Furthermore, we believe that APOS theory has a number of limitations, two of which we regard as especially important: 1) in APOS theory the construct 'object' is considered as the product of the encapsulation (reification) process. However, with this characterization, which basically comes from psychology, it is not clear how to address some issues related to mathematical objects, such as the nature of mathematical objects, their various types, the way in which they are formed and how they participate in mathematical activity.

In order to overcome this limitation it is helpful to consider the proposal of semiotic perspectives, especially the OSA, which regard mathematical objects as emerging out of mathematical practices. 2) The construct of 'semiotic medium' is not explicitly addressed by APOS theory, which does not specifically address the role of semiotic representations.

Recent research that has extended APOS theory through the incorporation of semiotic perspectives turns either to Duval's theory of semiotic registers (Trigueros et al., 2010) or to the OSA (Badillo, Azcárate & Font, 2010; Font, Montiel, Wilhelmi & Vidakovic, 2010). In line with Badillo, Azcárate and Font (2010) we consider that the OSA complements APOS theory as follows: 1) it extends APOS theory by specifically addressing the role of semiotic representations; 2) it improves the genetic decomposition by incorporating the ideas of semiotic complexity, network of semiotic functions, and semiotic conflicts; and 3) it offers a more detailed notion of mathematical objects due to the way in which it considers the nature of such objects and their emergence out of mathematical practices.

Objects in the CSM

The group of grounding metaphors includes the ontological type, where we find the object metaphor. The object metaphor is a conceptual metaphor that has its origins in our experiences with physical objects and enables the interpretation of events, activities, emotions and ideas, etc. as if they were real entities with properties. This type of metaphor is combined with other ontological, classical metaphors such as that of the "container" and that of the "part-whole". The combination of these types leads to the interpretation of ideas and concepts, etc., as entities that are part of other entities and which are constituted by them.

Ontological metaphors are considered as a group of metaphors that result from the projection of image schemas (container, whole-part, object, etc.) which, in our view, share a 'common territory'; therefore, there may be a certain hierarchy among them.

One question that remains open (Santibáñez, 2000) concerns the relationship between these image schemas. For example, one could consider that the object image schema is the fundamental schema and that the others are derived from it, or alternatively, that there is some other schema from which all (or some) of those mentioned are derived, for instance, the entity schema (Quinn, 1991) or the notion of thing (Langacker, 1998).

The link between the OSA and the CSM

In line with the onto-semiotic approach we consider that the process through which mathematical objects emerge from mathematical practices is highly complex (see section 3.1). Therefore, we believe that Lakoff and Nuñez's methodology of "mathematical idea analysis" is very important in terms of explaining the emergence of mathematical objects, but that it is insufficient to describe adequately this emergence and the nature of mathematical objects. This limitation was pointed out by various authors in the discussions that followed the publication of Lakoff and Núñez's book (e.g. Sinclair & Schiralli, 2003).

The way in which the OSA explains the emergence of mathematical objects not only extends and improves upon the explanation offered by the CSM, but also provides clarification of one of the central processes considered by the latter, namely metaphorical processes (Acevedo, 2008; Malaspina & Font, 2010; Font, Montiel, Wilhelmi & Vidakovic, 2010). The reasons for this are set out below.

Here we are interested in observing metaphorical processes from the 'unitary/systemic' duality proposed in the OSA, since the reification/decomposing processes in the OSA are associated with this unitary/systemic facet or dimension. When a mathematical abstraction is treated as an object, this is equivalent to adopting a unitary point of view on this object. On the other hand, the mathematical object can be treated from a systemic viewpoint, considering the actions that a subject can make on it and on the other objects, parts or processes that compose it. In the work of Lakoff and Núñez (2000), the unitary/systemic duality has a central role. On the one hand, the metaphor is elementary (A is B). However, the metaphor allows us to generate a new system of practices (systemic perspective) as a result of our understanding of the target domain in terms of the source domain. The CSM develops the elementary/systemic duality for different metaphors, a good example of which is the object metaphor (Font, Bolite & Acevedo, 2010):

Unitary: "Mathematical objects are physical objects"

Systemic: "Table 1. Metaphor projection"

In fact, most research on metaphors has been mainly targeted at studying such a duality. In other words, given a metaphor, the source and target domains are decomposed to determine what concepts, properties, relationships, etc. from the source domain are transferred to the target domain. The systemic vision of a metaphor leads us to understand it as a generator of new practices.

Source domain: Object image schema	Target domain: Mathematics
Physical object	Mathematical object
Physical objects are manipulated, found, discovered, etc.	Mathematical objects are manipulated, found, discovered, etc.
Physical objects are different from their material representations (for example, a clock is different from the drawing of a clock)	Mathematical representations are different from the mathematical objects they represent
Properties of the physical object	Properties of the mathematical object
Physical objects exist	Mathematical objects exist

Table 1. "Mathematical objects are physical objects"

The OSA considers that in order to carry out a mathematical practice the agent must have the basic knowledge required to do so. If we consider the components of the knowledge that the agent must have in order to develop and evaluate the practice that enables a problem to be solved (e.g., propose and solve a system of two equations with two unknowns), we can see that certain verbal (e.g., solution) and symbolic (e.g., x) language must be used. This language is the ostensive part of a series of concepts (e.g., equation), propositions (e.g., if the same term is added to the two sides of an equation, an equivalent equation is obtained) and procedures (e.g., solution by substitution) that will be used in making arguments so as to decide if the simple actions that make up the practice (where this is understood to be a compound action) are satisfactory. Hence when an agent carries out and evaluates a mathematical practice, it is necessary that it activates some (or all) of the elements mentioned above: situation-problems, language, concepts, propositions, procedures and arguments. By articulating these types of objects we obtain the configuration (hexagon in Figure 1). If, in addition to the "structure", it is necessary to analyze the genesis and functioning of the mathematical activity, other tools are necessary, especially some of the processes shown in Figure 1, as well as the metacognitive processes. The epistemic configuration tool allows us to see the structure of those objects that make mathematical practice possible and which regulate it within a specific institutional framework.

Because the OSA considers that, among other aspects, an epistemic/cognitive configuration (depending on whether the adopted point of view is institutional or personal) has to be activated in order to perform mathematical practices, and that the systemic vision of the metaphor leads us to understand it as a generator of new practices, it is natural to ask ourselves the following question: How is the metaphor related to the building components of epistemic/cognitive configurations? The conclusion drawn by Acevedo (2008) on linking metaphors, after he had studied in detail the linking metaphor used by Descartes when solving the problem of Pappus within the framework of analytic geometry, is that a linking metaphor projects an epistemic/cognitive configuration onto another one.

The epistemic/cognitive configuration construct allows us to explain and make precise the structure that is projected onto the linking metaphors. There is a source domain that has the structure of an epistemic/cognitive configuration (whether the adopted point of view is institutional or personal) and which projects itself onto a target domain that also has the structure of an epistemic/cognitive configuration. This way of understanding the preservation of the metaphoric projection improves upon the explanation of such a preservation given by Lakoff and Núñez (2000), who simply give a two-column table in which properties and concepts are mixed. The reader can intuit that the properties are projected onto properties and the concepts onto concepts.

The question which remains to be resolved is: what structure is projected in the case of a grounding metaphor? We believe that unlike in the case of linking metaphors, only some parts of the epistemic/cognitive configuration are projected. The specific study of each grounding metaphor will allow the identification of these parts.

FINAL CONSIDERATIONS

A sound theoretical understanding of mathematical objects must be a key part of any research on mathematics learning. The way in which the OSA understands the emergence of mathematical objects enables us to explain: (1) how primary objects emerge from configurations through mathematical practices; and (2) why a descriptive/realist view of mathematics is usually presented in mathematics classrooms. This account goes beyond the explanations offered by APOS theory and the CSM regarding the emergence of mathematical objects, and shows that what the latter two approaches do is highlight partial aspects of the complex process through which such objects emerge out of mathematical practices.

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