In England and Wales, mental mathematics has been emphasised in the primary school curriculum for over a decade but little has been done to prepare trainee teachers faced with the task of delivering this curriculum. The Williams Review (2008) recommends that this is an area which should receive ‘careful attention’ on initial teacher training programmes. Design-based research sought to use conceptual pedagogy to develop an effective programme for trainee teachers to develop their mental mathematics knowledge for teaching. This paper describes the use of the empty number line within the intervention programme for 129 trainee teachers on a one year post graduate training programme.

INTRODUCTION

The mathematics curriculum in England and Wales has changed dramatically over the last ten years since the introduction of the National Numeracy Strategy (NNS) in 1998. Pupils in primary schools are expected to be competent in adding and subtracting any pair of two digit numbers before they are taught formal written calculation strategies. As a result, mental calculation has attained a dominant position in the primary curriculum and teacher education programmes in the UK have required adaptation to reflect these changes.

Teachers’ Subject knowledge

Until the 1970s, little research had been carried out to determine what sort of mathematics teaching made for effective pupil progress (Ball, 1990). However, during the next few decades, mathematics education theorists and researchers began to consider the implications of Shulman's (1986) general categories of subject knowledge for the specification of the mathematics knowledge requirements for teaching and the construct was appropriated and refined for the purpose of mathematics research and practice.

In attending to mathematics content for teaching, some researchers concentrated on subject matter knowledge and delved deeply into the nature and purpose of the discipline to illuminate the task of ascertaining the mathematics knowledge required of teachers of mathematics (Goulding, Rowland & Barber, 2002). Further research was inspired by respect for the discipline of mathematics. It reflected research studies conducted in the USA, for example Ma (1999) who was specifically concerned with discipline-specific knowledge for teaching mathematics in its content (substantive and syntactical) and pedagogical forms, and in the UK, for example

Ma’s work (1999) explored the relationship between discipline-specific knowledge for teaching and pedagogy. Her results showed that American teachers had limited knowledge of mathematics and that their knowledge was often faulty. These findings were not new to American researchers (Ball, 1990) but what was novel was the comparison with Chinese teachers. This and other similar examples led Ma (1999) to contend that Chinese teachers identified and encouraged the specific conceptual understanding which was of importance in covering these types of problems. She described it as a profound understanding of fundamental mathematics (PUFM) – teachers’ recognition, within their teaching of:

- the connectedness of the simple but powerful ideas at the core of mathematics;
- the capacity of these ideas to sustain multiple perspectives and flexibility;
- the location of the ideas within a coherent and holistic disciplinary structure.

Crucially, within this specification, there was no differentiation between conceptual subject matter knowledge and pedagogical content (or specialist) knowledge. The conceptual foundation for PUFM was teachers’ sense of the connectedness of mathematics ideas; in pedagogical terms it meant that they formed knowledge packages around a central core. Conceptually, teachers had an awareness of the importance of simple but powerful ideas, which were especially stressed and developed through pedagogy. Conceptually, teachers were able to entertain multiple perspectives; pedagogically this meant that, as teachers, they could analyze their advantages and disadvantages and lead students to a flexible understanding. The coherence of their teaching over time reflected teachers’ conceptual sense of the primary mathematics curriculum as a whole. Pedagogically, it meant that they could exploit what pupils had already studied and create foundations for what was to follow.

**Mental mathematics**

A new curriculum for mathematics was established in England and Wales in 1998 (DfEE, 1998) in order to raise mathematical achievement (Brown, Askew, Baker, Denvir & Millet, 1998). This new curriculum – the National Numeracy Strategy (NNS) emphasised the development of mental calculation, which was in line with research in mathematics education in the Netherlands, where mental calculation and, in particular, informal strategies, had been a key component of the Realistic Mathematics Education (RME) programme.

The hundred-square, commonly found in primary classrooms in England and Wales, and used in the Netherlands prior to the introduction of the Realistic Mathematics Education programme, was seen as restricting, particularly in terms of mental
calculation. The Empty Number Line (ENL) was introduced as a model in schools in the Netherlands, which helped pupils visualise the ‘quantity value’ of numbers. It replaced the practice of partitioning using base ten that encouraged pupils to concentrate on the ‘column value’ (Thompson, 1999) of numbers. Beishuizen (1999) emphasised the benefits of the approach taken in the Netherlands in terms of general mathematics competence:

‘dealing with whole numbers supports pupils’ understanding and insight into number and number operations much more than the early introduction of vertical algorithms dealing with isolated digits’ (p.159).

Furthermore, Klein, Beishuizen and Treffers (1998) suggested that other benefits accrued from the action of replacing the hundred-square with the ENL, such as 'enhancing the flexibility of mental strategies' (p.427).

**Mental Mathematics and Teachers**

The new mathematics curriculum left university teacher educators charged with the need to make provision within their courses and programmes for opportunities to promote and assess the development of trainee teachers’ mental mathematics teaching competence.

So what were the knowledge requirements of teachers training to teach mental mathematics in primary schools? What did an understanding of mental mathematics mean for a primary trainee teacher charged with developing pupils’ ‘with the head’ (Beishuizen, 1997) use of mental mathematics, as opposed to the ‘in the head’ notion of rote learning? These problems were formulated into the following research questions: What are the most effective interventions to enable trainee teachers to enhance their mental mathematics subject knowledge for teaching and how can these be implemented during the university-based element in a one-year postgraduate primary teacher education programme?

**THE PROJECT**

The project involved the formulation of an innovative intervention programme to develop mental mathematics for teaching. Design-based research, through various diverse methods, involves accumulate a body of evidence that supports and enriches the theoretical principles underpinning a specific intervention and leads to the refinement of the intervention in situ. In their definition of design studies Wang and Hannafin (2005) argue that design-based studies provided access to:

...a systematic but flexible methodology aimed to improve educational practices through iterative analysis, design, development and implementation (p. 6).

The purpose of design-based research is the development of contextually sensitive theories in real-world settings and for these reasons, they offer a methodological
option which met the specific needs of this study manifested in its research question and underlying commitment to theory development.

**Theoretical principles for designing the intervention**

The design study was predicated on the belief that, in the context of teacher education, a knowledge-framed curriculum was an appropriate object for the study and development of trainee teachers’ mental mathematics subject knowledge for teaching. Furthermore, the knowledge-framed curriculum was driven by two theoretical principles, one discipline specific, relating to the conceptual foundations of mental mathematics and the other context specific, relating to the embedding of the conceptual within teaching.

The first principle was that the conceptual foundations of mental mathematics knowledge for teaching implied a commitment to working with numbers as wholes, an understanding of the reciprocal nature of mathematical operations as well as a strategic approach to calculation.

The second theoretical principle was that, in order to extend and realise these conceptual features within their teaching, trainee teachers required access, through conceptual pedagogy - the integration of conceptual and pedagogic knowledge into a bigger whole - to a connected, flexible, coherent approach to subject knowledge for teaching. In other words, the course needed to make provision for the development of what Ma (1999) described as profound knowledge.

The design of curricular activities in the study was based on design principles created by the contextualisation of these theoretical principles within the context of teacher education. Activities were designed to:

- Connect the three conceptual features – whole numbers, interchangeable operations and strategic approaches; provide the means to generate data for review of the effectiveness of the activities by inviting trainees to model pupil behaviour
- Focus on the simple and powerful ideas of mathematics (such as number structure, the laws of arithmetic, principles of counting, equations); re-visit and provide opportunities for their identification
- Nurture flexibility through comparisons by trainees, individually and in groups, with others’ perspectives, with their own previously held views and habits and through the application of mathematical ideas within discussions of ‘best’ strategies
- Reinforce the coherence of mental mathematics in relation to the primary curriculum

The process of design of activities and materials for use with a cohort of 129 primary trainee teachers during a year-long programme of teacher education was based on
these principles. Micro-analyses of trainee teachers’ interactions with the designed activities - individually, through group work within the university and in teaching placement contexts – led to review and modification of activities while offering a theoretically-informed commentary on their effectiveness.

Two strategies were adopted to improve validity and reliability: the use of data collection methods that captured the complexity of trainees’ interactions with materials and activities in as full a range of contexts as possible; and the creation of a partnership between a small group of trainees and the researcher as a means to challenge tacitly held assumptions and to establish consensus.

**The intervention**

During the programme, interventions, both planned and reactive, gave recognition to the importance of the principles outlined above. Each of them permeated the mathematics programme’s interventions, and specific activities were also planned to target particular areas.

The principles underlying the simultaneous presentation of the four number operations (addition, subtraction, multiplication and division) were derived from the requirements of mental mathematics – the reciprocity of the operations – and of conceptual pedagogy – in this case, connectedness and coherence.

Materials were prepared to highlight the way in which the empty number line could be used in teaching to model and, therefore connect, each of the four number operations. The intention during the initial stages of the session was to provide a stimulus for discussion within small groups in the form of: an explanation of the structure of the empty number line; a brief description of its introduction and use in the Netherlands; and an account of the way in which it was incorporated into the mathematics curriculum in the UK. At this stage group activity would involve the further exploration of the concept and the framing of questions for whole group discussion.

A variety of problems were chosen for use during the next stages of the session, with the specific aim of connecting the four operations. Once again, trainees were to be asked to participate in a group activity and to use the empty number line to find solutions to the problems. Pre-designed materials included the use of the empty number line to tackle, for example, division problems through the repeated subtraction of ‘chunks’. Finally trainees were to be asked, in a group activity, to reflect on the appropriateness of the empty number line in the context of their own preparation for teaching and to reveal their thinking.

During the year, qualitative data on trainees’ responses were gathered continuously during normal teaching sessions. This was achieved by creating opportunities for trainees to record the detail of the processes used individually and during group discussions and, where appropriate, to justify their responses in writing or verbally, in the whole group public arena.
Each mathematics session in the programme was planned to focus on progressing trainees’ mental mathematics knowledge for teaching by providing access, through conceptual pedagogy, to a flexible, connected and coherent approach to subject knowledge for teaching. This involved a commitment to working with whole numbers, the relationship between mathematical operations and a strategic approach to calculation over a number of weeks and environments (university and school placement) during the one-year programme.

Following the ENL activities described above, some trainees queried the effectiveness of the use of the empty number line. They were directed to research about alternative approaches to partitioning when calculating and critiques of these approaches (Thompson, 1999). After considering the research, trainees in six seminar groups, each of between 25 – 30 trainees, discussed the possible effects of focusing on one approach rather than the other. Initially, three groups reported in the feedback sessions that some members could not understand the need for such a tool because they had learnt strategies for each of the operations which ‘worked for them’. One group reported that this tool could become too much of a prop for pupils, encouraging counting in ones. For another group, the notion of using a number line was alien, as the experience of group members, prior to the course had tended to focus on a more formal approach to calculation. One member, commented that ‘the empty number line is an interesting concept, but perhaps one of the most challenging methods for me to grasp’. It also became apparent that some trainees were developing an algorithmic approach when using the empty number line. During workshop sessions where trainees were looking at subtraction by counting back or counting on, one trainee commented ‘I’m confused, which numbers do I write on the line?’ The response by other trainees was that, ‘you put the largest number on the right, then the other number at the other end’.

Such data led to the adaption of the intervention programme. Specific examples were introduced which allowed trainees to consider whether counting on or counting back was more appropriate. Trainees commented that specific examples also made them realise the importance of stopping to consider the numbers involved, which enabled them to assess the most appropriate calculation approach. The intervention was adapted further to provide opportunities for trainees to make use of individual whiteboards (similar to those used in primary schools) in order to do rough jottings using the empty number line. One trainee noted that the process of actually using the empty number line for her own calculations had enabled her to become more ‘flexible and confident in the use of the empty number line for teaching’.

The comment made, by one trainee, that she had found the empty number line to be a ‘revelation’ in the classroom, generated overwhelming agreement. The following comments are representative of the group’s views about the usefulness of the empty number line:

Trainee 1: The empty number line is a very important visual resource for pupils.
Trainee 2: I agree... that drawing jumps on the line works well as a natural way of keeping track and recording mental solution steps.

Trainee 1 described a scenario from her classroom, where a group of pupils were using the rounding and adjusting strategy. The example was 98 + 137. The pupils rounded the 98 to 100, but were then unsure whether to add or subtract the 2 which had been rounded. She demonstrated to the group how she had used the empty number line to encourage the pupils to consider the whole numbers involved. She reported that the pupils went on to complete the examples successfully, and claimed that although the pupils did not continue to draw the empty number line, they said that they were visualising it. Trainee 1 was challenged about whether the method had become an algorithm, but she was confident that the empty number line was used in quite a different way, as a means for visualising the relative sizes of the numbers.

At this stage the opportunity to broaden the cohort’s discussion was used to compare the whole numbers approach with alternative approaches in the form of formal written strategies or algorithms. A particular example was selected from an audit which had been used at the beginning of the year (199 + 174). Data from this audit revealed that over half of the cohort had used a formal written strategy at the beginning of the year. A number of the cohort described this as an unacceptable use of a formal written strategy. A quick paper survey was undertaken to establish how many of the cohort agreed with this ‘unacceptable’ use of the formal strategy and it was found that 93% of the group (106 trainees) agreed with this description for this particular item. Some trainees reported that they had not considered an alternative strategy at the beginning of the programme because they were using what they described as their security blanket - those algorithms which were quick to administer and gave reliable results. One group admitted that its members had not considered the numbers involved before diving in to carry out a procedure. A few trainees from other groups said that they were reluctant to move away from the strategies with which they felt comfortable. They could not appreciate why pupils should be introduced to so many different strategies as they felt this could cause confusion. All other groups, however, concluded that while formal algorithms may be effective and, ultimately, be the most efficient strategy, the ‘stop and think’ approach would encourage thinking about the numbers involved, before making a strategy choice and that this was of greater importance. This reference to a ‘stop and think’ approach was re-visited throughout the programme in order for trainees to consider alternative approaches.

The empty number line was further developed during discussion regarding problems involving decimals, where trainees reported that they made use of the empty number line for these types of problems purely as a visual image, to consider the relative sizes of the numbers involved. In this context, one trainee noted that the empty number line had helped her to appreciate the structure of the number system. She wondered why she had been ‘so concerned about working with decimals, because if
you just think about them on the number line, they are just normal numbers, but zoomed in’. The commitment to focusing on numbers as wholes was an indication of the success of the use of the ENL as a model for developing mental mathematics for teaching.

CONCLUSIONS

As noted above, some trainees were initially sceptical about the value of the empty number line, but for many, this perception changed over the course of the intervention. After spending time in the classroom, trainees began to change their views as they saw the potential benefit of this approach. One trainee observed, during school experience, what was seen as the benefit of planning activities to emphasise connectedness using the ENL ‘these children were slowly beginning to make some creative connections between numbers that they could use to help them solve a variety of calculations.’

Although most trainees concluded that a flexible approach to calculation was what they would strive for in their teaching, this commitment was not always evident in their own calculations. The following is one trainee's explanation of her initial response to 199 + 174, where she had used a formal written method:

‘When I look down I think 199 and I know automatically that you should see this as 200.... But when it is written down like that it’s not, to me, it’s not obvious that it would relate to 200....I definitely only see the digits.’

During review the data which had been collected throughout the programme was referred to and used to provide some insight into these contrasting experiences. A minority of trainees had routinely reported during the first workshops that they found the standard formal written methods efficient and straightforward to apply, and that it was difficult to understand why any other methods were taught in school. When invited to reflect on their experiences during seminars and in schools, there was general agreement that it felt as if they were ‘learning backwards’.

Such comments indicate the potential of conceptual pedagogy for change. Even in a generally negative example there is evidence that a trainee was enabled to analyse and identify a lack of coherence as the source of the insecurity of her knowledge of mental mathematics:

‘See this is the problem that I have got, I’ve got all these little facts and they are all over the place. So that, so somehow I know that obviously three quarters is 0.75 and then you times it by 10 and it’s 75, I’ve got all these useless, they are not useless, but, well, they are useless unless I can apply them.....See that’s the thing, I’ve got all these jumbled up facts and I know all these silly little things, that that needs to be that but I don’t know why, there is no sense to it.’

This clearly showed that there had been insufficient time for this particular trainee to come to terms with her early experience of mental mathematics, which was
sufficiently strong to resist the coherence promoted in the programme. However, the statement shows that the trainee does appreciate the need for integration and for meaning in the knowledge possessed by teachers of mental mathematics.

Others had been able to use experience of the programme to come to terms with their own knowledge base. One trainee explained how the programme had affected her mathematics knowledge:

'Like most students I would instinctively choose the column method and (abbreviated) long division for these problems, but I would have great trouble teaching them and this brought home the requirements that I must in a sense ignore my own knowledge and capabilities.'

This comment is yet further evidence of the recognition that teachers have a different requirement in terms of mathematics knowledge. It is not sufficient to know mathematics, what is significant is the way in which mathematics is known for teaching. These types of comments suggested that, even if they were not fully successful in terms of their own facility with mental mathematics, trainees were able to appreciate the significance of the different approach to mental mathematics that is required of teachers.

Although some trainees had not found a way to resolve their deficiencies in their own mental mathematics through engagement in the programme, for the overwhelming majority of participants, who claimed that they now recognised the negative and lasting effects of embedded routines and practices learnt while they were at school, the situation was acceptable, since they argued that they could operate differently in their teaching. Further speculation about the intellectual significance of the way trainees dealt with the dissonance created by their own development would not be warranted on the basis of this study’s data. However, the study has provided legitimate grounds for seeking answers to questions about the limits, durability and architecture of this way of holding knowledge.

At this stage it is possible to conclude that the evolving behaviours of trainees fuelled the development and refinement of an intervention that was based on conceptual pedagogy as applied within the context of mental mathematics knowledge for teaching. Successful outcomes include the illumination and exemplification of existing theoretical constructs such as coherence and connectedness over a range of mental mathematics content involving the use of the empty number line.

**REFERENCES**


