SPECIAL EDUCATION STUDENTS’ ABILITY IN SOLVING SUBTRACTION PROBLEMS UP TO 100 BY ADDITION

Marjolijn Peltenburg & Marja van den Heuvel-Panhuizen
Freudenthal Institute for Science and Mathematics Education,
Utrecht University, the Netherlands

In this study we examined special education students’ use of indirect addition for solving two-digit subtraction problems. Fifty-six students (8- to 12-year-olds), with a mathematical level of end Grade 2, did an ICT-based test on subtraction. Although most students had not been taught indirect addition they frequently applied this procedure spontaneously. For about two-thirds of the problems that have an adding-on context and for about half of all the problems with a small difference between the minuend and subtrahend indirect addition was used. The main prompt for using indirect addition were the item characteristics. Indirect addition was identified as a highly successful procedure for special education students and the best predictor of a correct answer was found in combination with a stringing strategy.

Keywords: Special education, Indirect addition, Information and Communication Technology (ICT), Assessment, Empty number line

INTRODUCTION

At the end of primary school many special education (SE) students are considerably behind on the topic of subtraction with numbers up to 100 compared to their peers in regular education (Kraemer, Van der Schoot, & Van Rijn, 2009). To improve the achievements of SE students in solving subtraction problems, it is suggested to teach them one particular way of solving calculations (see e.g., Milo & Ruijssenaars, 2002; National Mathematics Advisory Panel, 2008).

There are several reasons for challenging this advice. Firstly, the idea of teaching only one method goes against the goal of developing numeracy in students. This goal implies that students should be able to choose a suitable strategy when solving number problems (see e.g., Van den Heuvel-Panhuizen, 2001; Warry, Galbraith, Carss, Grice, & Endean, 1992). Secondly, teaching one method implies that for solving particular problems students may have to follow an unnecessarily long way to come to an answer (see e.g., Torbeyns, Ghesquière, & Verschaffel, 2009). Thirdly, using prescribed methods can lead to ‘didactical ballast’ (Van den Heuvel-Panhuizen, 1986) for students. This means that students have to become skilled at following the given recipes which is not always easy for them, because the ownership is completely on the side of the teacher or textbook author.

Despite of these disadvantages, the idea of teaching students with mathematical difficulties one solution method is nowadays still often advocated. This plea results from the assumption that weak learners do not have the necessary insights to choose an approach that fits to a particular task (see e.g., Milo & Ruijssenaars, 2002;
In the study reported in this paper the tenability of this claim is investigated by means of an Information and Communication Technology (ICT)-based assessment. The focus of the study is on using an addition procedure for solving subtraction problems up to 100.

**Strategies and procedures for solving subtraction problems**

For solving addition and subtraction problems with numbers up to 100 generally three different types of strategies can be distinguished: splitting, stringing, and varying (Van den Heuvel-Panhuizen, 2001). These idealized strategies of which examples are given in Figure 1 have in common that they describe *how we deal with the numbers involved* (in splitting both numbers are decomposed in tens and ones, in stringing one number is kept as a whole number, and in varying one or both numbers are changed in order to get an easier problem).

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<td>Indirect addition</td>
<td>64−52 = 62−58 +2 *</td>
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<td>Multiple operations</td>
<td>77−29 = (77+1)−(29+1)=48</td>
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* This way of solving this problem is not very common

Figure 1. Relation between procedures and strategies illustrated with problems

A different way of describing a calculation is by focusing on *how the operation is carried out*. From this perspective, two main procedures for solving subtraction problems can be distinguished: (1) direct subtraction (DS), which means taking away the subtrahend from the minuend (e.g., solving 62−58= _ by 62−50=12; 12−2=10 and finally 10−6=4), and (2) indirect addition (IA), which means adding on from the subtrahend until the minuend is reached (e.g., solving 62−58= _ by 58+2=60 and 62+2=62). Together the strategies and the procedures offer a complete framework for
describing how students solve subtractions up to 100. See Figure 1. DS is likely to go together with splitting or stringing. For IA and IS, stringing is the most obvious strategy, although splitting can be applied as well. Finally, when a varying strategy is applied multiple operations are required.

**Solving subtraction problems by indirect addition**

In connection to the earlier mentioned chasm of ideas about whether or not teaching one solution method to students who are weak in mathematics there is also debate on whether SE students are able to flexibly solve subtraction problems up to 100 by applying an IA procedure. For example, a few recent intervention studies have revealed that even students in regular primary education have great difficulty to incorporate IA for solving subtraction problems up to 100 (De Smedt, Torbeyns, Stassens, Ghesquière, & Verschaffel, 2010; Torbeyns, De Smedt, Ghesquière, & Verschaffel, 2009).

However, these studies are contradicted by other intervention studies that do support the claim that already in the first grades of primary mathematics education, students of different ability levels in mathematics can learn to flexibly solve subtraction problems by applying IA (Blöte, Van der Burg, & Klein, 2001; Menne, 2001).

**Factors influencing students’ procedure use**

Important factors that may influence students’ procedure use when solving a subtraction problem are: (1) student characteristics, such as their general mathematics level, (see e.g., Torbeyns, De Smedt et al., 2009), (2) teaching characteristics, for example, whether or not students’ have been taught a particular procedure (see e.g., Menne, 2001), and (3) problem characteristics. With respect to the latter, the influences of the following three features of a subtraction problem are discussed: (a) the numbers involved, (b) the problem format (context problems or bare number problems), and (c) the available auxiliary tools.

**Influence of the numbers involved**

Several studies (e.g., De Smedt et al., 2010; Torbeyns, De Smedt et al., 2009) have indicated that subtraction problems that require crossing the ten and have a small difference between the minuend and subtrahend may evoke the use of IA. However, IA may also be an efficient procedure in solving large-difference problems with a small difference around the tens and requiring crossing the ten. For example, 82–29= _ may be easily solved by IA (i.e., 29+1=30; 30+50=80 and 80+2=83, so 1+50+2=53). Finally, research suggested that small-difference problems that do not require crossing the ten (e.g., 47–43 = _) may also evoke the use of IA (Gravemeijer et al., 1993).

**Influence of the problem format**

Two didactical phenomenological interpretations of subtraction are: (1) subtraction as taking away, and (2) as determining the difference. In the first interpretation, the matching operation is that of taking away the subtrahend from the minuend, whereas
in the second interpretation bridging the difference between the subtrahend and minuend by adding on is also an option. Both interpretations need to be addressed if we want students to learn subtraction in a more complete way (Freudenthal, 1983; Van den Heuvel-Panhuizen & Treffers, 2009).

To contribute to this broad understanding of subtraction students should not only be presented bare number problems. Different studies (e.g., De Smedt et al., 2010; Torbeyns, De Smedt et al., 2009) revealed that bare number problems hardly evoke the use of IA. Context problems, on the contrary, have the possibility to open up both interpretations of subtraction (Van den Heuvel-Panhuizen, 2005).

**Influence of the auxiliary tools**

To support students in carrying out calculation problems up to 100, different models can be used. Basically, two main models can be distinguished: group models and line models (Van den Heuvel-Panhuizen, 2001). Group model, such as rods of ten and blocks of one, are particularly adequate to represent a splitting strategy together with DS. Line model, such as the empty number line, are mostly suitable to support a stringing strategy either in combination with DS or IA. The empty number line thus has the possibility to represent both interpretations of subtraction: taking away by jumping backwards and adding on by jumping forwards.

**The present study**

The present study was set up to investigate whether and under which conditions SE students are able to use IA for solving subtraction problems up to 100, and whether they can solve subtraction problems correctly when applying this procedure. The purpose of the study was to clarify the role of the numbers involved, the format of the problem (context or bare number problems), the presence of a digital empty number line as an optional auxiliary tool, and the occurrence of prior instruction in IA. The study has two foci (I) students’ spontaneous use of IA, i.e., applying IA without being asked to use this procedure, and (II) students’ success rate when applying IA.

**METHOD**

**Participants**

In total, 56 students from fourteen second-grade classes in three Dutch SE schools participated in the study. The participating students (39 boys, 17 girls) were 8 to 12 years old, with a mean age of 10 years and 6 months (SD=10.4 months). All students had a mathematical ability of level C or lower at the CITO Monitoring Test for Mathematics End Grade 2 (Janssen, Scheltens, & Kraemer, 2005).

**Materials**

**ICT-based test on subtraction problems**

An ICT-based test was developed that contains a collection of items in which item characteristics are varied systematically. These characteristics include number characteristics and format characteristics.
The number characteristics refer to the size of the difference between the minuend and subtrahend (small means <7 or large means >11), whether or not the tens have to be crossed (e.g., 61–59=_), and whether or not the minuend and the subtrahend are close to the ten (<3). The format characteristics refer to whether or not the items are presented as a bare number problem (BN) or as a context problem. The latter can describe a taking-away situation (ConTA) or an adding-on situation (ConAO). Figure 2 shows a screen shot of one of the context problems that reflects an adding on situation.

![Train ticket item](image)

**Figure 2. Train ticket item:** the accompanying read aloud instruction is: “A train ticket costs 41 euro. Father has already paid 29 euro. How many more Euros need to be paid?”

The ICT-based test is compartmentalized into two parts. The first 15 items do not feature the number line tool, whereas the last 15 items do. This digital empty number line operates by touch-screen technology.

After a short introduction, the students worked individually on a touch-screen notebook. Students were said that they are completely free in choosing a particular solution method. Apart from giving an answer they also had to report verbally how they solved the items. The students’ working on the screen was recorded by means of Camtasia Studio software.

**Online teacher questionnaire**

To collect data about the students’ prior instruction on subtraction problems an online teacher questionnaire was developed. The link of the questionnaire was sent by email to the fourteen teachers who are responsible for teaching mathematics to the students that participated in the study. All fourteen teachers filled in and submitted the questionnaire. The questionnaire contains two questions on the topic of ‘subtraction
up to 100’, which were meant to collect data about (1) the models and materials the teachers use for teaching subtraction up to 100, and (2) the procedures (DS and/or IA) they teach their students for solving subtraction problems up to 100.

RESULTS
In the analysis of the data we included all the cases in which the students gave an answer to an item. Of the 1680 possible cases (56 students doing all 30 items each) 147 cases were missing. This resulted in 1533 cases to be analyzed. DS and IA were clearly the most frequently applied procedures. DS was applied in 63% of the total cases and went together almost equally often with a stringing and splitting strategy. IA was applied in 32% of the total cases; in almost 90% IA was applied in combination with a stringing strategy.

Different conditions and SE students’ spontaneous IA use

Numbers involved
IA was most frequently applied in small-difference problems, i.e., in 50% of the 322 cases involving items with crossing the ten and in 43% of the 324 cases involving items without crossing the ten. DS was most frequently applied in large-difference problems, i.e., in 90% of the 282 cases involving items with crossing the ten, in 78% of the 306 cases involving items without crossing the ten, and in 66% of the 299 cases involving items with a small difference around the tens and requiring crossing the ten.

Problem format
We found that an adding-on context generally goes together with IA, whereas a taking-away context mostly resulted in a DS procedure. That is, IA was applied in 68% of the 509 cases involving an adding-on context and DS was applied in 75% of the 510 cases involving a taking-away context. When solving bare number problems the students also had a strong preference for DS. That is, in 91% of the 514 cases involving bare number problems, DS was applied.

Prior instruction
The teachers’ responses to the online questionnaire revealed that two different textbook series were used in the fourteen classes. Although these textbook series each contain some missing addend problems, they do not explicitly address the inverse relation between addition and subtraction. Because teachers could have paid attention to IA without it being addressed in the textbook series we also asked them which procedures they taught their students for solving subtraction problems. Their answers made clear that all teachers taught DS. Only three teachers responded that they have taught both DS and IA. This means that, in total, 15 students were taught both procedures. These students applied IA in 32% of the total 419 cases they had solved; the students who were not taught IA applied this procedure in 32% of the total 1114 solved cases.
Use of the empty number line

According to the data that were retrieved from the online teacher questionnaire all students were familiar with the empty number line for doing subtraction. In 15 out of the 30 items, the students had available the optional digital empty number line for solving the subtraction problems. The 15 items resulted in 778 cases of processed items. In 131 of these cases the empty number line was actually used for finding an answer. The students used IA in 15% of these 131 cases. In the 647 cases in which the students saw the empty number line but did not use it, IA was applied in 33% of the cases.

Multilevel analysis with IA use as dependent variable

A cross-classified multilevel model was carried out with IA use as dependent variable. This analysis revealed, among others, that the random item effect (SD=2.41) was quite large compared to the random student effect (SD=.85), indicating that IA use is mainly an item characteristic. This means that the application of IA is more strongly elicited by the nature of an item than by the specific preference of a student, which implies that students applied IA in a flexible, item specific way.

SE students’ success rate in IA and DS use

In 70% of the 489 cases in which the students applied IA their answers were correct. In the 976 cases in which they applied DS their answers were correct in only 48% of the cases.

Different conditions and success rate in IA and DS use

Numbers involved

Small-difference subtraction problems were solved with the highest success rate when IA was applied. Of the 162 cases involving items that require crossing the tens, the students solved 86% correctly with IA and of the 139 cases that do not require crossing the tens, the students solved 88% correctly with IA. When students applied DS, the highest percentage of correct answers was found in small-difference subtraction problems without crossing the ten. Of the 176 cases involving such items, the students solved 67% correctly by DS.

Problem format

In all three problem formats (ConAO, ConTA and BN) the students solved more problems correctly than incorrectly when they applied IA. The highest percentage of correct answers was found in solving items that reflect taking away, i.e., 82% of the 108 cases involving taking-away items were solved correctly. When using DS, we found that in all three problems format the students solved about half of the items involved correctly.

When not taking into account the used procedure and comparing the three problem formats, then students appeared to be most successful in solving context problems
that reflect adding on (ConAO) and least successful in solving bare number problems (BN).

*Other conditions*

The students who had received IA instruction solved 76% correctly of the 134 total cases they had solved by IA. The students who did not receive IA instruction solved 68% correctly of the 355 total cases they had solved by IA. Moreover, the IA-instructed students solved 54% correctly of the 270 total cases they had solved with DS. The students who did not receive IA instruction solved 45% correctly of the 706 total cases they had solved with DS.

Of the 778 cases in which the empty number line was available in the items IA was applied in 235 cases. In 20 cases the students actually used the number line and in 215 cases they did not. It appeared that the percentages of correct answers in both groups were about the same, namely 75% and 73% respectively.

**Multilevel analysis with success rate as dependent variable**

A cross-classified multilevel model was carried out with success rate use as dependent variable. This analysis was aimed to reveal the influence of strategy use and procedure use on students’ success rate in applying IA. It was found that only the use of a stringing strategy increased success rate significantly ($b=.52$, SE=.19, $p < .05$). The best predictor of a correct answer appeared to be the combination of a stringing strategy together with the IA procedure ($b=.96$, SE=.37, $p < .05$).

**CONCLUSIONS**

Our study was limited in scope and therefore further research is needed with more students covering more schools. Moreover, we did not carry out a detailed inventory of the students’ prior instruction in IA. Therefore, information on the quality of the instruction was missing. This might explain why we did not find any influence of prior instruction on the students’ success rate in applying IA.

In general, more student characteristics and more details about their prior instruction should be taken into account to acquire a deeper understanding of SE students’ potential in solving subtraction problems. Nevertheless, the present study has revealed three striking outcomes:

1. SE students are able to use IA spontaneously, i.e., without being asked to do so.
2. SE students are rather flexible in applying IA to solve subtraction problems.
3. SE students are quite successful when solving subtraction problems by IA.

We think these findings plea for a reconsideration of the approach to mathematics education in SE which advocates only teaching the straightforward taking-away procedure. Such an approach clearly underestimates SE students’ mathematical ability. Finally, this study has shown that solely focusing on strategies (splitting, stringing, and varying) or solely on procedures (DS and IA) is a too restricted way of
investigating students’ ability to solve number problems. Both should be taken into account as our study showed that the best predictor of a correct answer is the combination of IA and stringing.

NOTE
¹ The ICT-based test was developed by the authors of this paper and programmed by Barrie Kersbergen, a software engineer at the Freudenthal Institute.

REFERENCES


