A GAP BETWEEN LEARNING AND TEACHING GEOMETRY
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This work originates from a worksheet, presented to 9 years old pupils at the end of an activity related to a teaching experiment on geometrical understanding in primary school (Marchini, Vighi, 2011a). The analysis of the protocols surprised us: we found a big variety of reasoning and solutions, but especially many mistakes and misconceptions. Therefore we decided to study in depth the pupils’ behaviours which emerge from the protocols. In particular, we analyse the possible roots of wrong geometrical reasoning with the aim to investigate on the nature of the gap between teaching and learning geometry.

KEYS words: geometrical misconceptions, teaching practice, figures.

THEORETICAL FRAMEWORK

“Shape is a fundamental construct in cognitive development in and beyond geometry” (Lester, 2007) but its role is often underestimated or neglected. The role of semiotic representations in teaching and learning of geometry is studied in depth from Duval (2006):

“The special epistemological situation of mathematics compared to other fields of knowledge leads to bestow upon semiotic representation a fundamental role. First of all they are the only way to access mathematical objects”.

In other words, the impossibility of a direct approach with mathematical concepts compels the teacher to furnish representations to present and to explain it. Following Santi & Sbaragli (2007), these representations can take place of mathematical objects and they

“can lead the student to consider valid ‘parasitical information’ bound to the specific representation, in contrast with the generality of the concept”.

In particular, concepts of perimeter and area in geometry teaching in Italian primary schools are introduced with pupils 8-11 years old, but usually only measuring work is done. Measuring length can be realized by using a ruler, measurement of areas is more complex, since “the use of a ruler in measuring area is indirect” (Murphy, 2010), but it can be made using other artefact, for instance squared paper. Research shows that

“…the early introduction of measurement in geometry presents the risk of confusing a g-quantity with its measure (q-quantity), whereas it is in fact necessary to emphasise that an object has a length or a surface, even if these are not measured” (Marchetti et al., 2006).

Stressing too early the measure of length by abusing of simplicity and power connected with artefact ‘ruler’ might foster the idea that everything is measurable
with this tool and furthermore that a geometrical task requires always a computation on the numbers gotten by the ruler, starting from a suitable formula. In this way we can give also the idea that ruler is an instrument for procedural competence, but that “falls short in helping children develop an understanding of space” (Nitabach & Lehrer, 1996) and it could bring children to confuse area and perimeter (Dickson et al., 1984). This attitude can be ascribed also to the pre-eminence of the additive conceptual camp (Vergnaud, 1990), but we can consider other reasons of this mismatch. When a pupil “... has to differentiate on a physical object itself or on a geometrical representation, between the $g$-quantities in either one or two dimensions” (Jaquet, 2000), an obstacle inevitably arises. This is the “perimeter-area conflict”.

Without a measurement we can speak of length and of extension as $g$-quantity, but neither of distance, nor of area as $q$-quantities. Nevertheless we can compare (with the relation of equality or of congruence, equi-extension, and also with ordering) lengths and extensions and make a small arithmetic part, proportions and the equality of proportions.

We suggest that the learning of geometry is helped by working on comparison between two, or more, $g$-quantities related to carefully chosen shapes. Moreover it may be that, before than geometry had become institutionalised as a school topic, pupil is able to compare the areas of two surfaces, for example by superimposing the sheets of paper on which they are drawn, but not be able to calculate the area. Similarly, s/he may be able to compare the lengths of two curved lines, but not be able to calculate their measures.

Following Santi & Sbaragli (2007), we can think that the use of ruler brings an ‘unavoidable misconceptions’, i.e. a misconception that “do not depend directly on the teacher’s didactic transposition” (whereas the ‘avoidable misconceptions’ “depend exactly on the didactic choices performed by the teacher”).

Some other misconceptions are reported in the literature. Lunzer (1968) presents notion of ‘false conservation’. Stavy and Tirosh (1996) articulated this suggestion as the intuitive rule ‘more A, more B’, which, in case of geometrical shapes, is the assumption that as the perimeter increases so the area will increase. Alternatively the intuitive rule can be presented as ‘same A, same B’ i.e. the same perimeter will mean the same area (a conflict perimeter vs. area). In our experience the ‘same A, same B’ misconception is applied also in the reverse sense: from equality of area to equality of perimeter (a conflict area vs. perimeter).

Another possible cause of the conflict perimeter-area might be the use of the squared paper. It is well known that “... the counting of squares on grids provides more success and may represent the notion of ‘space filling’” (Murphy, 2010), which is the ultimate idea underlying the concept of surface, but it has at least other two contraindications. Dickson et al. (1984) focused on the dichotomy discrete-continuous. Surface is an example of continuous $g$-quantity, from Aristotle on, and the measure of a similar entity requires the continuity in the numerical field in which the number (area) is
placed. The superimposition of a discrete setting to a continuous entity requires cautious actions. The didactic practice of considering ‘well-mannered’ numbers for the measure of a rectangle sides could ingenerate the idea that discrete is enough for Geometry, but this idea collapses with circle.

The second contraindication resides in the fact that both perimeter and area are computed by counting squares. But often teachers give instruction of the type: “Draw on your exercise-book a horizontal segment which is three squares long”. Thence square, which is the unit of measure for area became also the unit of measure for length. The counting all for perimeter is resumed by a sum, thence the same model is applied to area. Therefore the difficulties induced by squared paper can be considered as unavoidable misconceptions, in the meaning of Santi & Sbaragli (2007).

In early grades we assist to the blind application of formulae. Literature suggests that children use them without a deep understanding of their meaning. This behaviour is evident when pupils do not take care of the success of their answers (Dickson, 1989), with mistakes revealing that children have difficulty in “interpreting the physical meaning of the numerical representation of area” (Zacharos, 2006). The studies of Pesek & Kirshner (2000) suggested that the early teaching of formulae presented ‘interference of prior learning’; Zacharos (2006) speaks about an ‘instructive obstacles’.

Houdement and Kuzniak (2006) introduced in geometrical approaches three paradigms to explain various purpose aimed by Geometry. In particular, they put the problem of the transition from Geometry I (“Natural Geometry with source of validation closely related to intuition and reality with eventually the use of measurement and or construction by real tools) to Geometry II (“Natural and axiomatic Geometry base on hypothetical deductive laws related to a set of axioms close as possible on the sensory reality) (Kuzniak et al., 2007).

THE RESEARCH AND THE TASK

This paper is suggested from a worksheet, in Figure 1, which is part of a more wide research about the effects of the learning and practice of isometries on the standard approach to geometry (Marchini & Vighi, 2011a). We refer to this work for a more detailed a-priori analysis, for methodology and also for statistical data about the experimentation. The research involved an experimental sample (ES, No. 38) and a control sample (CS, No. 33), in grade 4 (9-10 years old) at the end of the school year. The task allows different strategies of solution: we analysed and discussed the results of this text by the light of the presence/absence of a reasoning making use of the isometries (Marchini & Vighi, 2011a). The pupils’ answers made so clear the gap between learning and teaching topics such as perimeter and area, but not only these. Global success was very low; so we wonder about it and we decided to devote a specific paper to our observations.
Calculate perimeter and area of these shapes (measures are in centimetres). Then explain your solution.

![Shapes 1 and 2]

Calculate the area of the following shape.

![Shape 3]

**Figure 1. The task [2]**

**TASK ANALYSIS, RESULTS AND REMARKS**

Protocols analysis suggested us some relevant issues about teaching and learning of geometrical topics. For each one, instead of presenting in succession the three topics of the paragraph title, we decided to show a synopsis of these three: the first (a) contains suggestions about teaching practice, the second (b) is related with the a-priori analysis of our worksheet, the third (c) concerns the results obtained and their interpretation.

**Which is the role of words in geometry?**

(a) The (Italian) curriculum asks teacher to present simple shapes and their names. So, the names become labels for figures, often using a classification by exclusion, with the effect that a square is not a rectangle.

(b) The text of our worksheet uses few words; we write only the questions and some information about the units of measures. Some numerical data are written near to the sides of the shapes. It is an unusual presentation of a geometrical problem in (Italian) school.

(c) Pupils understood without difficulties the text and the meaning of ‘shape’, ‘perimeter’ and ‘area’.

**Which is the role of the shape in the traditional problems of geometry?**

(a) The role of the shape is often marginal, since the objects of geometrical problems are well known figures, as squares, rectangles and so on. The first practice with school problems is only to translate a text in suitable operations: it happens also in the case of geometrical problems. In particular, the representation of a geometrical entity by a drawing is a visual tool supporting the text of a problem, therefore both teacher and textbook in their drawings could not take care of respecting numerical data present in the problem. Moreover usually a geometrical problem involves only one figure.
(b) In our worksheet, shapes are not traditional and they play a fundamental role, since they contain the majority of geometric and numerical information. They perplex pupils and they force them to think in geometrical way. In the Shape₁ children have the possibility to come back to a more familiar setting, by decomposing the shape in two rectangles, with the help of one ‘horizontal’ segment or by two ‘vertical’ segments. It is also possible to decompose Shape₂ in rectangles, but this procedure is less evident and it induces to carry out other strategies. Shape₃ is not a polygon, it can be refused even if semicircle is a known figure.

(c) Teachers told us that at the beginning our ‘strange’ shapes disconcerted pupils, but in a second time some of them found in the rectangle the organizing concept for solving the task.

What ‘perimeter’ means for children?

(a) Often perimeter is reduced to a calculation, using some formulas, without geometrical meaning. For the rectangle, they are \( p = (b + h) \times 2 \) or \( p = 2b + 2h \). In fact, pupils are requested to make only some substitutions of letters with numbers and to compute. In this way, geometrical aspect of the problem is neglected and the arithmetical register of representation prevails.

(b) Our Shape₁ and Shape₂ have respectively 8 and 12 sides, they needs a ‘long’ addition of several numbers to calculate their perimeter. Shape₃ is not a polygon, its perimeter cannot be obtained with known formulas; for this reason, we decided to ask its area only.

(c) The more frequent wrong procedure used for perimeters consists in adding only the given numbers, without searching the unknown data (36.61% for Shape₁ and 40.85% for Shape₂). The exclusive use of given data can arise from a didactical obstacle: usually the problem contains exclusively the necessary numbers and it needs to use all of them. We consider that the prevalence of arithmetical register and the lack of perimeter meaning lead to mistake. For instance, some children decide to divide Shape₁ in two rectangles aiming at to work with customary figures, after they sum both perimeters to obtain the perimeter of whole shape, counting twice the partial ‘ghost’ side.

A misconception (Santi & Sbaragli, 2007) probably induced from our work about isometries in the classes involved in the original experiment is this: before children made a translation for ‘turning’ the shape into a rectangle and after they calculate the perimeter; in other words, translation shall preserve perimeters. We can consider this also an example of the ‘same A, same B’ misconception (Stavy & Tirosh, 1966), with the interpolation of the reasoning with isometries. This behaviour suggests a new conflict: the application of local isometries gives a ‘true’ rectangle whose area is equal to the area of the initial shape, but perimeter not. In fact the rectangular final shape is used to find the perimeter after the local transformation [3]. So, instead of using the perimeter procedure for computing area (perimeter-area conflict) we notice the use of a procedure for area to compute perimeter. We can consider it as ‘area vs.
perimeter conflict’. The regularity of transformed figure could be another reason for applying this procedure (Gestalt).

**What ‘area’ means for children?**

(a) In Italian school the starting activities about area often consist in a subdivision of shapes in squares and in counting them. In a second time there is the passage from a square-unit to the cm$^2$ as unit measure. This passage from an arbitrary unit measure to the conventional one is difficult, thence it requires a slow and accurate transition. The idea of area as measure of surface slips the mind, while the retained suggestion is related to calculation or to the count of the number of squares.

The curricula suggest to start with decomposition of different simple shapes and with the recognition of their possible equi-decomposability, but usually in school this activity has a marginal role. Therefore pupils are not supported in understanding that different shapes can be transformed in a same shape, only with some operation of ‘cut and paste’.

(b) In our worksheet this idea to ‘cut and paste’ could be ‘winning’, since it is possible to obtain easily only one rectangle starting from Shapes 1, 2 and 3, having the same area of them. Shape$_1$ is not a rectangle, but two numbers are placed in the ‘canonical’ positions as the measures of ‘basis’ (4.5) and ‘height’ (5.0) of a rectangle. The same graphical configuration is present in Shape$_2$ and the multiplication of two suitable numbers would give the right result. The possibility to obtain a rectangle with the same area would be more evident in Shape 3; it is also motivated from the impossibility to use formulas for calculate semicircle area.

(c) In fact, for many of children the shapes are deeply different. The setting of the worksheet suggested to someone (15%) the computation of Area$_1$ by the multiplication of 4.5$\times$5.0, using a right interpretation for area, but an inadequate procedure since is the area of only a part of the shape. The total rate of right answers in the case of Area$_1$ is less than 10%. Our ‘strange’ shapes suggested someone the idea to use the bigger rectangle which can be inscribed in the shape or the minimal rectangle which include the given shape and to apply to it the classical formula for area. A protocol shows the calculations of 6.5$\times$5 for Shape$_1$, and of 4.5$\times$7 for Shape$_2$; so, the main idea is to find “by quantifying how far it is between the endpoints of the object” (Lester, 2007) and to came back to a more usual shape.

The need to find a way to know area of Shape$_3$ suggested the idea to transform the shape in a rectangle and in the same time the possibility to use the same procedure for the first and the second shape; so, some children came back and they change the results obtained previous in a different way. We notice that few children calculate only one area, observing that the others are the same. There are also children who calculate the three areas without comparing results. This can be a consequence of the lack of control individuated by Dickson (1989) and Zacharos (2006).

Decomposition in rectangles is used in Shape$_1$ and Shape$_2$. Some pupils (50% in ES and 15% in CS) use different decompositions of shapes and they indicate it using
arrows. In several cases it doesn’t imply that the measures of surfaces are the same: there is a gap between geometric and arithmetic meaning. In particular, about Shape 3, geometrical intuition isn’t supported from the arithmetic one: a child recognises the possibility to transform Shape 3 in a rectangle, he writes: “I cannot find the area since the measures is different at all”. Sometimes after the calculations a geometrical reasoning arises. In a girl’s protocol we found “Why these areas are equal? Since the shapes are different but it is possible to transform both in the same rectangle”.

**Which is the role of ruler and of measures?**

(a) At the beginning, in traditional Italian school ruler is used as support to draw straight segments, in grade 3 it becomes an instrument for measure. Measurements of linear \( g \)-quantities and relative units are introduced early, before geometrical concepts. When geometrical figures are presented, measures are used to calculate their perimeters. In particular, for the rectangle even if we give only two measures and we write them respectively near to two consecutive sides of the shape, it is implicit that the measures of the other two sides are known. Also the use of the same words, ‘basis’ and ‘height’, for congruent and parallel segments reinforces this idea.

(b) In our worksheet some measures are missing. There is also implicit information: some segments are parallel and congruent. Using it, in Shape 1 it is possible to obtain missing numbers from the known measures of some segments. In Shape 2 it is required a more complex reasoning: the measure of an ‘horizontal segment’ can be achieved by a subtraction \( 4.5 - (1.0 + 2.0) \), another ‘horizontal segment’ measures 0.5 as its parallel segment, the measure of ‘vertical’ segment can be obtained again with subtraction \( 6.5 - (3.0 + 1.5) \) or using the idea of roto-translation of the upper rectangle. Another possibility is to see rectangles “with eyes of the mind” and to use the well known geometrical property that in a rectangle the opposite sides are parallel and congruent.

(c) A child writes: “To find an omitted data, I searched the other sides that look like the mine”, so he doesn’t use direct measure but congruence of segments. To denote it, another child uses the locution “in front”. Both show the presence of Geometry paradigm II (Houdement & Kuzniak, 2006). A third pupil remains in GI since he doesn’t perceive this congruence and he measures two congruent segments obtaining 1.7 cm and 1.8 cm. For him this discrepancy is not problematic.

Some pupils decided to overcome the problem of omitted data using a graduate ruler. Some pupils use both the given number written in the worksheet and the number founded with ruler. In this way they misunderstood the role of unit of measure. So, the shape stops to represent a geometrical figure and it becomes the real object that must to be measured. The given and the obtained numbers are considered as ‘pure’ numbers and measures become ‘absolute’. Others children decide to measure again all sides and to use only the numbers obtained: in this way they measure the perimeters of figures similar to Shape 1 or Shape 2. This behaviour is typical of the Geometry paradigm I.
A boy obtains two different perimeters for the same shape: firstly in a correct way he find the omitted data and he sums all the measures of lengths; afterwards he measures all the sides with rulers and sum the numbers obtained in this way. So, the main idea is to perform an addition and not to exploit the concept of perimeter. It is caused by the sliding from the geometric to the arithmetic field.

**Which is the role of squared paper in traditional geometry?**

(a) During the lessons of mathematics in Italian school usually we use exercise-books with squared paper. It allows to draw shapes and to copy them when the teacher draws to the blackboard and also to “count squares” (obviously in different ways) to calculate perimeters and areas.

(b) In our worksheet we chose do not use squared paper, but the tradition way in which the textbooks present shapes.

(c) The lack of data induced some child to copy Shape\textsubscript{1} and Shape\textsubscript{2} on squared paper as a way of bringing back the task to a more customary environment. We observe different kinds of copy. The first is based on the use of transparent sheets, with superimposition of squared paper on the worksheet, obtaining a congruent shape. The second way consists in the use a squared paper with side of squares of 0.5 cm and to report on it the correct measures, obtaining a similar shape. A third possibility is that the copy on squared paper is a rough representation of the shape, without respecting its proportions, with wrong local scales. We notice also that during the reproduction on squared paper, some child did not care to a congruent or similar reproduction, but only to preservation of angles.

Sometimes shapes are embedded in a grid, with squares or with rectangles whose aim is at measuring perimeter or area or both. Since the sides of squares have arbitrary measures this gave completely different results, for incomprehension of the unit of measure, identified with one square. In the same setting there are protocols showing a rough approximation, since small or great parts of a square is assessed as a whole square or as an its half.

**CONCLUSIONS**

The evidence of experiment shows a very wide gap between the pupil’s learning and the teacher’s purpose. Among the multiple factors which can be held responsible of this situation, we focus on the role of the shape in teaching and learning of geometry. Shape is link between a concrete object and an ideal geometrical figure. Often in primary school shape looks like representation of real object, in secondary school it becomes representation of a geometrical figure. In this practice we see the transition from one paradigm to the other one: shape requires a good transfer from Geometry paradigm I to (at least) the II one. In our experiment we met both way of thinking, but the oddity of our issues induces a lot of pupil to seek ‘refuge’ in Geometry I. So shapes became objects: pupils decide to measure the sides in the drawings (or some of them) treating them as ‘real objects’. The same attitude is revealed by copies on squared paper and the counting of linear or two-dimensional squares.
It suggests that the usual convention related to a possible drawing in proportion that we used in the worksheet must be clarified from the beginning. Proportional reasoning must associate with the presentation of geometry: it is necessary to realise a shape ‘in proportion’ (Vighi, 2010). In this way shape achieves its role and also its usual meaning also for pupils in Primary school.

“Geometrical measurement can serve as a bridge between the two critical domains of geometry and numbers, with each providing conceptual support to the other” (Lester, 2007).

In fact, in our experimentation often we observed the lack of the bridge from these two realms. We can presume that the need of computing perimeters and areas of each one shape obscured the geometrical aspects: it blocked the comparison among shapes and it hindered in recognition of their equi-extension. From the other part, the very unusual geometrical aspect of the shapes leaded to renounce to any calculation or to make strange and senseless operations. We observed also a lot of arithmetic mistakes and misconceptions, but they cannot take place in this paper.

We have evidence that the practice of isometries helps the solving process.

NOTES

1. Work done in the sphere of Italian National Research Project Prin 2008PBBWNT at the Local Research Unit into Mathematics Education, Mathematics Department, Parma University, Italy.

2. In the original sheets the side which measure is indicated as 4.5 cm are 5.3 cm long and the other measures are in the same ratio.

3. We submitted the same task to in service young teachers and in some protocols we observed the same mistake.

REFERENCES


