TOWARDS A COMPREHENSIVE THEORETICAL MODEL OF STUDENTS’ GEOMETRICAL FIGURE UNDERSTANDING AND ITS RELATION WITH PROOF

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This research study examined the fit of various models regarding geometrical figure understanding and its relation with proof. Data were collected from 457 middle and high school students. Structural equation modelling affirmed the existence of nine first-order factors revealing the differential effect of perceptual and recognition abilities, the ways of figure modification, construction of a figure and proof. The three second-order factors which represented the perceptual, operative and sequential apprehension were regressed to a third-order factor that corresponded to the geometrical figure understanding. Results indicated that geometrical figure understanding has a strong effect on logic apprehension. Data analysis provided support for the invariance of this structure across the two educational levels.

INTRODUCTION AND THEORETICAL FRAMEWORK

Fischbein (1993) called geometrical figures “figural concepts” since these entities are simultaneously concepts and spatial representations. Generality, abstractness, lack of material substance and ideality reflect conceptual characteristics. A geometrical figure also possesses spatial properties like shape, location and magnitude. In this symbiosis, it is the figural facet that is the source of invention, while the conceptual side guarantees the logical consistency of the operations (Fischbein & Nachlieli, 1998). The double status of external representation in geometry often causes difficulties to students when dealing with geometrical problems due to the interactions between concepts and images in geometrical reasoning (e.g. Mesquita, 1998). Duval (1995, 1999) distinguishes four apprehensions for a “geometrical figure”: perceptual, sequential, discursive and operative. To function as a geometrical figure, a drawing must evoke perceptual apprehension and at least one of the other three. Each has its specific laws of organization and processing of the visual stimulus array. Particularly, perceptual apprehension refers to the recognition of a shape in a plane or in depth. In fact, one’s perception about what the figure shows is determined by figural organization laws and pictorial cues. Perceptual apprehension indicates the ability to name figures and the ability to recognize in the perceived figure several sub-figures. Sequential apprehension is required whenever one must construct a figure or describe its construction. The organization of the elementary figural units does not depend on perceptual laws and cues, but on technical constraints and on mathematical properties. Discursive apprehension is related with the fact that

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mathematical properties represented in a drawing cannot be determined through perceptual apprehension. In any geometrical representation the perceptual recognition of geometrical properties must remain under the control of statements (e.g. denomination, definition, primitive commands in a menu). However, it is through operative apprehension that we can get an insight to a problem solution when looking at a figure. Operative apprehension depends on the various ways of modifying a given figure: the mereologic, the optic and the place way. The mereologic way refer to the division of the whole given figure into parts of various shapes and the combination of them in another figure or sub-figures (reconfiguration), the optic way is when one made the figure larger or narrower, or slant, while the place way refer to its position or orientation variation. Each of these different modifications can be performed mentally or physically, through various operations.

Recently, some researchers (Deliyianni, Elia, Gagatsis, Monoyiou, & Panaoura, 2009; Elia, Gagatsis, Deliyianni, Monoyiou, & Michael, 2009) made an effort to verify empirically some of the cognitive processes underline the geometrical figure understanding proposed by Duval (1995, 1999). Elia et al. (2009) gave emphasis on the cognitive processes involved in operative apprehension. Furthermore, Deliyianni et al. (2009) affirmed the existence of a third-order model that involved six first-order factors indicating the differential effect of perceptual and recognition abilities, the ways of figure modification and measurement concept, three second-order factors representing perceptual, operative and discursive apprehension and a third-order factor that corresponded to the geometrical figure understanding. Both studies also suggested the invariance of this structure across elementary and secondary school students. Keeping in mind the underlying cognitive complexity of geometrical activity (Duval, 1995) and the transition problem from one educational level to another universally (Mullins & Irvin, 2000) the main aim of this research study was to confirm a comprehensive theoretical model concerning middle and high school students’ geometrical figure understanding which involves the whole spectrum of geometrical figure apprehension types, i.e. perceptual, discursive, sequential and operative apprehension and the relation between their corresponding cognitive processes. It should be mentioned that concerning discursive apprehension Harada, Gallou-Dumiel and Nohda’s (2000) conceptualization is used. Harada et al. (2000) indicated that the hypothetical-deductive proof is produced by this kind of apprehension. In fact, the discursive apprehension or logic apprehension, the term which is used in the present paper, is produced by inferences based on definitions and valid procedures of proof.

HYPOTHESES AND METHOD

In the present paper the following hypotheses were examined: (a) There is a relation between students’ geometrical figure understanding and their performance in proof tasks, (b) Perceptual, sequential and operative apprehension influence middle (grade 9) and high (grade 10) school students’ geometrical figure understanding, (c)
Perceptual and recognition abilities have a differential effect on perceptual apprehension, (d) The three ways of figure modification (i.e. merelogic, optic and place way) have a differential effect on operative apprehension, (e) The abilities to construct and describe a figure’s construction differentially affect sequential apprehension, (f) Inferences based on definition and procedures for proof differentially affect discursive (logic) apprehension, (g) There are similarities between middle and high school students in regard with the structure of their geometrical figure understanding and (h) Differences exist in the geometrical figure understanding performance of middle and high school students.

The study was conducted among 457 students, aged 15 to 16, of middle (Grade 9) and high (Grade 10) schools in Cyprus (252 in Grade 9, 205 Grade 10). Taking into account, Duval’s (1995, 1999) apprehensions for a “geometrical figure” the a priori analysis of the test (Appendix) that was constructed in order to examine the hypotheses of this study is the following:

1. The first group of tasks includes task 1 (PE1a, PE1b, PE1c, PE1d, PE1e, PE1f, PE1g), 2 (PE2a, PE2b, PE2c, PE2d, PE2e, PE2f) and 3 (PE3a, PE3b, PE3c). These tasks examine students’ perceptual apprehension of a geometrical figure. The task 1 examines students’ ability to name figures. The tasks 2 and 3 examine their ability to discriminate and recognize in the perceived figures several subfigures.

2. The second group of tasks includes task 4 (OP4), 5 (OP5), 6 (OP6), 7 (OP7), 8 (OP8) and 9 (OP9). These tasks examine students’ operative apprehension of a geometrical figure. The tasks 4 and 5 require a reconfiguration of a given figure, the tasks 6 and 7 an optic way of modification, while the tasks 8 and 9 demand the place way of modifying two given figures in a new one in order to be solved.

3. The third group of tasks consists of the tasks 10 (SE10), 11 (SE11), 12 (SE12) and 13 (SE13) that correspond to sequential figure apprehension. The tasks 10 and 11 require students to construct a figure, while the tasks 12 and 13 investigate students’ ability to describe the construction of a figure.

4. The fourth group of tasks includes the verbal problems 14 (LO14), 15 (LO15), 16 (LO16), 17 (LO17), 18 (LO18) and 19 (LO19) that correspond to logic apprehension. On the one hand, the verbal problems 14, 15, 16 and 17 demand inferences based on definitions in order to be solved. On the other hand, tasks 18 and 19 require inferences based on procedures for proof for their solution.

Right and wrong or no answers to the tasks were scored as 1 and 0, respectively. The results concerning students’ answers to the tasks were codified with PE, OP, SE and LO corresponding to perceptual, operative, sequential and logic apprehension (proof tasks), respectively, followed by the number indicating the exercise number.

Confirmatory factor analysis (CFA), by using the EQS program, was used to explore the hypotheses about the structural organization of the various dimensions
investigated here (Bentler, 1995). The tenability of a model can be determined by using the following measures of goodness-of-fit: \( x^2 \), CFI and RMSEA. The following values of the three indices are needed to hold true for supporting an adequate fit of the model: \( x^2 / df < 2 \), CFI > 0.9, RMSEA < 0.06. A multivariate analysis of variance (MANOVA) was also performed to examine if there were statistically significant differences between middle and high school students concerning their performance in the various dimensions of the figure understanding.

RESULTS

Confirmatory factor analysis model. A series of CFA models were tested and compared. Specifically, the first model involved only one first-order factor associated with all the tasks. This model was the most parsimonious, it disregarded though the related theory and past empirical work which pointed out that different cognitive processes are needed in order to solve: perceptual, operative, sequential and logic apprehension tasks. The fit of this model was poor [CFI= 0.52, \( x^2 \) (702)=3933.98, RMSEA= 0.10]. The second model that was constructed and tested involved four first-order factors corresponding to the perceptual, operative, sequential and discursive apprehension and one second-order factor on which all the first-order factors were regressed. A chi-square difference test indicated a significant improvement in fit between the first and the second model [\( \Delta x^2 \) (43) =1505.09, p<0.001] due to the second-order factor inclusion. However, the fit of the second model was also poor [CFI= 0.81, \( x^2 \) (459) = 1682.99, RMSEA= 0.08].

The third model took into account Deliyianni’s et al. (2009) findings and moved a step forward involving sequential apprehension dimension, the three ways of figure modification in operative apprehension dimension and the deductive reasoning dimension. A chi-square difference test indicated a significant improvement in fit between the second and the third model [\( \Delta x^2 \) (52) = 611.83, p<0.001]. Besides, the fit of the third model was acceptable [CFI= 0.91, \( x^2 \) (511) = 1071.162, RMSEA= 0.05]. Even though the third model fitted the data reasonably well, the need to confirm that this was the best fitting model arose. Taking into account that visualisation is thought to be useful to some aspects of mathematical proof (Hanna & Sidoli, 2007), a fourth model was tested. Its fit was acceptable [CFI= 0.94, \( x^2 \) (444) = 815.08, RMSEA= 0.04], as well. A chi-square difference test indicated a significant improvement in fit between the third and the fourth model [\( \Delta x^2 \) (67) = 256.08, p<0.001] due to the causal relation between geometrical figure understanding and logic apprehension inclusion. The first, second and third tested models are presented in Figure 1. Factor loadings are omitted.
Figure 1. The first, second and third CFA tested models

Figure 2 shows the results of the elaborated model, which fitted the data reasonably well. The first, second and third coefficients of each factor stand for the application of the model in the whole sample (Grade 9 and 10), middle (Grade 9) and high (Grade 10) school students, respectively. Particularly, the third-order model which is considered appropriate for interpreting geometrical figure understanding, involves nine first-order factors, four second-order factors and one third-order factor. The four second-order factors correspond to the geometrical figure perceptual (PEA), operative (OPA), sequential (SEQ) and logic (LOA) apprehension, respectively. Perceptual, operative and sequential apprehensions are regressed on a third-order factor that stands for the geometrical figure understanding (GFU). Therefore, it is suggested that the type of geometric figure apprehension does have an effect on geometrical figure understanding, verifying our second hypothesis. On the second-order factor that stands for perceptual apprehension the first-order factors F1 and F2 are regressed. The first-order factor F1 refers to the perceptual tasks, while the first-order factor F2 to the recognition tasks. Thus, the findings reveal that perceptual and recognition abilities have a differential effect on geometrical figure perceptual apprehension (hypothesis c). On the second-order factor that corresponds to operative apprehension the first-order factors F3, F4 and F5 are regressed. The first-order factor F3 consists of the tasks which require a modification of a given figure in a mereologic way. The tasks which demand an optic way of modifying a given figure compose the first-order factor F4 and the tasks demanding the place way of modifying two given figures in a new one in order to be solved constitute the first-order factor F5. Therefore the results indicate that the ways of figure modification have an effect on operative figure understanding (hypothesis d). The first-order factors F6 and F7 are regressed on the second-order factor that stands for sequential apprehension. The first-order factor F6 refers to the tasks which demand the construction of a figure, while the first-order
factor F7 consists of the tasks in which the description of a figure’s construction is needed. Thus, the results indicate that the two abilities differentially affect sequential apprehension (hypothosis e). According to the factor loadings, operative apprehension is more strongly related with geometrical figure understanding than perceptual and sequential apprehension.

On the second-order factor that stands for logic apprehension the first-order factors F8 and F9 are regressed. The first-order factor F8 refers to the tasks which require inferences based on definition, while the first-order factor F9 to the tasks which inferences based on processes of proof are needed. Thus, the findings reveal that the kind of inferences has a differential effect on this kind of apprehension (hypothesis f). Loadings indicate that geometrical figure understanding have a strong effect on logic apprehension (hypothesis a).

![Figure 1. The CFA model of the geometrical figure understanding in relation with proof processes.](image)

To test for possible similarities between the two educational levels concerning their geometrical figure understanding the proposed three-order factor model is validated for middle and high school students separately. The fit indices of the model tested for both middle $[x^2 (445) = 658.59, CFI= 0.94, RMSEA= 0.04]$ and high school students are acceptable $[x^2 (438) = 659.61, CFI= 0.94, RMSEA= 0.05]$. Thus, the results are in line with our hypothesis that the same geometrical figure understanding structure holds for both the middle and the high school students. It is noteworthy that some
factor loadings are higher in the group of the high school students suggesting that the specific structural organization potency increases across the ages. Besides, the factor loading in grade 10 regarding perceptual apprehension is lower than in grade 9, while the factor loading for sequential apprehension is higher than the corresponding in grade 9. This finding indicates that as students grow up are based more on mathematical properties and less on perceptual laws and cues.

*The effect of students’ educational level.* Table 1 presents the means and the standard deviations for perceptual, operative, sequential and logic apprehension in the two educational levels. Overall, the effect of students’ educational level is significant (Pillai’s $F_{(4, 452)}=7.03$, $p<0.001$). In particular, the mean value of high school students in geometrical figure perceptual apprehension (PEA) is statistically significant higher ($F_{(1,452)}=16.94$, $p<0.001$) than the mean value of middle school students. Similarly, the mean value of high school students in operative apprehension tasks (OPA) is statistically significant higher ($F_{(1,452)}=14.26$, $p<0.001$) than the mean value of middle school students. In the same way, the mean value of high school students’ performance in sequential apprehension tasks (SEA) is statistically significant higher in comparison with middle school students’ performance ($F_{(1,452)}=11.88$, $p<0.001$). Even though, the performance of high school students in logic apprehension tasks (LOA) is also higher than the performance of middle school students this difference is not statistically significant ($F_{(1,452)}=3.83$, $p=0.05$). Therefore, the findings verify the last hypothesis stating that differences exist in the performance of middle and high school students. In particular, high school students’ performance is higher in all the types of geometrical figure apprehension.

<table>
<thead>
<tr>
<th>Educational Level</th>
<th>PEA</th>
<th>OPA</th>
<th>SEA</th>
<th>LOA</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\bar{X}$</td>
<td>SD</td>
<td>$\bar{X}$</td>
<td>SD</td>
</tr>
<tr>
<td>Middle</td>
<td>0.78</td>
<td>0.21</td>
<td>0.59</td>
<td>0.23</td>
</tr>
<tr>
<td>High</td>
<td>0.86</td>
<td>0.20</td>
<td>0.67</td>
<td>0.24</td>
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</table>

Table 1: Means and standard deviations in the different dimensions of the geometrical figure apprehension for middle and high school students

**CONCLUSIONS**

This study examined the fit of various models regarding geometrical figure understanding and its relation with proof processes. Using structural equation modelling we constructed and verified a comprehensive model for geometrical figure understanding. Moving a step forward in relation with previous studies (e.g. Elia et al., 2009; Deliyianni et al., 2009) which verified Duval’s (1995, 1999) taxonomy, the proposed model involves the whole spectrum of geometrical figure apprehension types and the relation between their corresponding cognitive processes with cognitive processes concerning proof.
According to the results, the three second-order factors which stand for perceptual, operative and sequential apprehension are regressed to a third-order factor that corresponds to the geometrical figure understanding. Results suggest that geometrical figure understanding has a strong effect on students’ performance in proof tasks. This is in line with the findings of previous research studies (e.g. Hanna & Sidoli, 2007; Giaquinto, 2007) that suggested that there is a potential contribution of visual representations to mathematical proof. Findings reveal also that operative apprehension is the one which contributes the most to geometrical figure understanding. Taking into account that visualization consists only operative apprehension (Duval, 1999) the important role of this kind of apprehension confirms empirically Duval’s (1999) opinion that there is not understanding in geometry without visualization. The specific result indicates also that teaching and learning should give emphasis in this kind of apprehension since visualization is not primitive. In fact, the use of visualization requires specific training, specific to visualize each register (Duval, 1999). However, the model points out the important role of the other types of geometrical figure apprehension, as well, taking into account that even though coordination between them is needed each one is distinct from the other (Duval, 1999). Besides, findings affirmed the existence of nine first-order factors revealing the differential effect of perceptual and recognition abilities and the ways of figure modification, construction and proof. Thus, the results verified Duval’s (1995, 1999) and Harada’s et al. (2000) categorization, respectively.

In addition to extent our knowledge about students’ geometrical figure understanding, this study may give valuable information to curriculum designers and teachers of both middle and high school education. The elaborated model offers teachers a framework of students’ thinking while solving a wide range of geometrical tasks in a systematic manner within and between the two educational levels. Therefore, the proposed framework may be used as a tool in mathematics instruction and designing tasks on geometry in both middle and high school. The framework of this study appears to be useful from an assessment perspective, as well. It may provide teachers with valuable and specific information on students’ thinking in geometry based on prior knowledge and enable them to enhance this thinking by giving appropriate support through the tasks focused on the competences and cognitive processes for the geometrical figure understanding and the proof.

Concerning age, it is important to stress that the structure of the processes underlying the geometrical figure understanding in relation with proof processes was invariant across the two age groups tested here. These findings enhance the validity of the proposed framework and support its potential to coherently describe and predict students’ understanding in geometry irrespectively of their grade, even during the transitional phase from middle to high school. However, findings reveal differences between middle and high school students’ performance. In fact, the results provide evidence for the existence of three forms of elementary geometry, proposed by Houdement and Kuzniak (2003). We may assume that in this research study, middle
school teaching is mainly focused on Geometry I (Natural Geometry) that is closely linked to the perception, is enriched by the experiment and privileges self-evidence and construction. On the other hand, high school teaching gives emphasis to Geometry II (Natural Axiomatic Geometry) that it is closely linked to the figures and privileges the knowledge of properties and demonstration. As a result, in the case of middle school students geometrical figure is an object of study and of validation, while in the case of high school students geometrical figure supports reasoning and “figural concept” (Fischbein, 1993). However, the knowledge produced by quantitative research studies might be too abstract and general for direct application to specific local situations, contexts, and individuals. For this reason, further research is needed to evaluate the feasibility of using this framework for developing effective instructional programs for the teaching of geometry in regular classroom situations in middle and high education.

REFERENCES


**APPENDIX**

1. Name the squares in the given figure:

2. Recognize the figures in the parenthesis (KEZL, IEZU, EZUM, IUOJ, LGJ, BEI).

3. Underline the right sentence:
   a. Fig. A has equal perimeter with Fig. B
   b. Fig. A has smaller perimeter than Fig. B
   c. Fig. A has bigger perimeter than Fig. B

4. Fans is looking the box 1 and 2 in the horizon. He says that the boat 1 has exactly the same size with box 2. Is his opinion right? Explain your answer.

5. Underline the right sentence:
   a. Fig. A has equal area with Fig. B
   b. Fig. A has smaller area than Fig. B
   c. Fig. A has bigger area than Fig. B

6. a. Draw an arc AB with centre C, equal to the arc MQ with centre D. (SE10)
   b. Describe the construction of the figure.

7. The points M, N and P are the midpoints of the sides of triangle ABC. Show that the quadrilaterals AFMN, BMDP, and GNPM are parallelograms. (LO14)

8. In the figure below:
   - AB and AC are equal
   - Line (e) is parallel to BC
   - AH is perpendicular to BC

   Underline the right sentence:
   The length of NH
   a) is smaller than NH
   b) is equal to NH
   c) is bigger than NH
   d) it cannot be determined (LO18)