This paper focuses on primary school teachers’ intuitive approaches to uncertain circumstances. Investigating how perspective teachers handle probabilistic situations, in fact, may shed light on how teachers’ training in probability would influence pupils’ learning of probability. To achieve this goal, in an exploratory study, data from a basic undergraduate course in probability for preservice primary school teachers were gathered and examined through semiotic lenses. A cognitive model is presented and discussed and an example is shown.

**Key-words:** elementary probability, preservice teachers, intuition, semiotics.

**INTRODUCTION**

It is well acknowledged that human beings have often to deal with uncertainty and they may judge, decide and behave under the sole guidance of their intuitions. Along with their intuitive reasoning, human beings may represent the situation somehow in their mind. The relation between the representations people use and their intuitive reaction in uncertain situations is the focus of this study. Within this perspective, there is a special sample of human beings that plays a crucial role: trainee primary school teachers. Teachers, in fact, are in charge of planning, carrying out and evaluating the classroom activities. Hence, the analysis of their intuitive approaches to probability when using representations – during their training – is a relevant issue.

This paper is an attempt to construct a cognitive model to frame probabilistic reasoning within intuition and semiotic perspectives. The first section is devoted to the construction of the model according to the following steps: firstly, the meaning of intuitions and semiotic representations is characterized with respect to probabilistic thinking; secondly, three cognitive levels at which the estimation of uncertainty occurs are sketched; thirdly, the approaches at each level are presented and discussed also with respect to other existing results. The second section describes methodological issues and the third one shows the results and a possible interpretation through the cognitive model. Some concluding remarks end the paper.

**THE COGNITIVE MODEL AND ITS THEORETICAL BACKGROUND**

**Intuitions and semiotic representations**

Intuitions in the process of learning probability have been a fruitful branch of research in Psychology and in Mathematics Education. In 1970s the psychologists Kahneman and Tversky (1972; Tversky & Kahneman, 1973), started to focus on the psychology of prediction and probability judgments on one side, and Fischbein (1975) on the other side highlighted the role of intuitions in probability and
combinatorial concepts. In 1980s, a rich literature in Mathematics Education concentrated on (children’s) probabilistic intuitions. Fischbein and Gazit (1984) claimed that “little is known about probabilistic intuitions and their development under the influence of systematic instruction” (p.1). Hawkins and Kapadia (1984) proposed a synthesis about the key questions concerning children’s understanding of probability. On these footsteps, in the subsequent years a number of researches concentrated on this topic: for example, Konold et al. (1993) studied students’ inconsistent reasoning about different aspects of the same situation. This kind of researches focused on the students’ approaches to uncertain situations, and consequently helps to frame and analyse students’ misconceptions and intuitions that can arise and be driven by such different approaches. In this paper, the term ‘intuition’ is used in opposition to the logical knowledge, according to Fischbein:

“the process of thinking is composed of two basic interwoven aspects: the logical, analytical, discursive one (the evolution of which, in children and adolescents, has been studied by Piaget and his co-workers), and intuitive cognitions characterized by self-evidence, immediacy, globality, coerciveness.” (Fischbein, 1998, p.1)

Now, a few words need to be said about the semiotic approach. Semiotics is a powerful tool for interpreting didactical phenomena. As Ernest points out

Beyond the traditional psychological concentration on mental structures and functions ‘inside’ an individual [semiotics] considers the personal appropriation of signs by persons within their social context of learning and signing. Beyond the behavioral performance this viewpoint also concerns patterns of sign use and production, including individual creativity in sign use, and the underlying social rules, meanings and contexts of sign use as internalized and deployed by individuals.” (Ernest, 2006, p.68)

The way teachers use representation are analyzed from Duval’s perspective (Duval, 2008): it identifies mathematical thinking and learning with the coordination of semiotic system according to the following operations: treatment (transforming a representation into another within the same semiotic system), and conversion (transforming a representation into another in another semiotic system). My contention is that the coordination of different semiotic systems support/constrain intuitions in probabilistic reasoning. Depending also on the task, the semiotic resources employed, and the background of the individual, I have singled out three levels at which people can refer when estimating the probability of an event.

Three levels to estimate probability: experience, discrete quantities, theory

The first level to deal with is the level of the experience. At this level dice are rolled, playing cards are randomly selected, coins are flipped etcetera. And at this level people win or lose. There is a huge literature in Mathematics Education focusing on and discussing the role of the context in mathematics tasks, the relationships between experience and beliefs, the goal of mathematics to prepare for citizenship, and so on (van der Kooij, 2010). In this background, which is still the focus of several researches all round the world, I take into account the experience and its role in
probability learning processes for two main reasons. The first reason is that probability – as a modeling activity and more than other mathematics domains such as algebra – has a special relationship with the experience. Outcomes in probability, in fact, often refer to real objects. Moreover, the frequency of such outcomes has a relationship with the probability estimate of an event (Piattelli-Palmarini, 1995). The other reason is that – just for its special relationship with the world of experience – the probability and its learning result strictly intertwined with intuitions and misconceptions (Fischbein & Gazit, 1984).

At the experience level, people have to estimate the probability of the outcomes. Hence, people start to represent somehow in their mind the situation: the representations would be perceptive and approximate, and they serve the purpose of counting the number of, say, possible and favorable events. Sometimes people (and, above all, teachers when teaching) represent the events using less perceptive representations such as arrows, Venn diagrams, tables, etcetera. When using representations that help counting, people are no more at the experience level, but stay at the second level: the level of arithmetic thinking. This is the level at which people shift from perception and approximation to quantities. Such quantities, at this level, may be integer numbers, and become ratios or percentages.

There is a third level, the level of theory. At this level, the formulas, the axioms and the theorems of probability lie. The representations are formal and abstract. However, in some tasks there is no need to (know and) stay at this level, but to correctly solve the task proper use of representations at the arithmetic level is enough. Outside the level of theory, in fact, it is possible to introduce probabilistic tools in a more immediate way. At each one of the three levels, in fact, it is possible to accomplish an estimate of uncertain situations, following different approaches.

**Different approaches to uncertain situations**

At the level of experience, estimating the probability of an event means avoiding the use of quantities. Semiotically speaking, it entails the use of mainly graphic and pictorial registers, related to perceptive and sensorial experience. I claim that people in general – and young children in particular – are able to say that an event is nearly impossible or almost certain, without involving any sort of computation. A confirmation of my claim has been given by Jeong, Levine & Huttenlocher (2007), who examined whether children are able to reason about proportions in the context of continuous amounts before they are able to do so in the context of discrete sets. The tasks they proposed used a graphic register. Figure 1 shows an example: in that task, children had not to use any sort of number, but were able to say which figure had the largest area. Jeong’s and his colleagues’ interpretation of children’s better performance in the continuous than in the discrete conditions was that

“It seems likely that they used a perceptual strategy similar to that used by infants and young children to code the extent of one length or region in relation to another […]. For example, children may have compared target and target areas in each donut [part–part
coding] [...], or have compared the red area in each donut to the total area of each donut [part–whole coding].” (Jeong, Levine & Huttenlocher, 2007, p. 252)

![Figure 1: one of Jeong, Levine and Huttenlocher (2007) tasks: donuts. Which donut has the greater percentage of its area in grey?](image)

In order to characterize it, two keywords related to the approach at the level of experience may be: ‘perception’ and ‘approximation’. Moreover, it is characterized by self-evidence, immediacy, globality, and coerciveness, entailed by graphical/pictorial representations. The experience level can be considered intuitive, since such characteristics – accomplished by graphical/pictorial representations – are the features that define intuition mentioned above, according to Fischbein (1998).

The concept of proportion is, according to Fischbein (1998), another form of intuitive thinking. I claim that this kind of approach belongs to the level of arithmetic, since it is necessary to come out the level of experience and quantify the situation to deal with it. In this case, immediacy and globality are characterized by internalized operational schemas, conveyed by arithmetical representations. At the level of arithmetic, moreover, I distinguish two approaches: one is more related to the experience level and consists in using only percentages and proportions; the other one is more linked to the level of theory and involves the use of ratios. To better define the percentage approach, I refer also to Huerta & Lonjedo (2007), who describe the mostly arithmetical type of thinking as

“students think in quantities but they recognize events and their associated frequencies or percentages.” (Huerta & Lonjedo, 2007, p. 735)

Hence, this level is characterized by arithmetic representations and proportional schema: on the one hand frequencies/percentages, on the other one ratios/fractions.

The approach at the level of theory regards probability as a function that assigns a number between 0 and 1 to an event. It overcomes operational thinking and, through symbolic registers, it is characterized by a functional and relational reasoning. At this level, intuition is the result of a specific mathematical training. It allows students to deal with more complex situations: for example, Lecoutre and Durand (1988) showed that using ratios solely may induce misconceptions in estimating the probability of composite events. Other examples in literature concern misconceptions in Bayesian thinking (for example Falk, Falk & Levin, 1980).

**Relations with the existing literature**

I have shown the three levels at which judging under uncertainty may occur. With respect to the aforementioned approaches, Hawkins and Kapadia (1984)
distinguished four definitions for probability, and labeled them as: (1) ‘a priori’, (2) frequentist, (3) subjective and intuitive, and (4) formal probabilities.

The level of experience is similar to the definition provided by Hawkins and Kapadia for subjective probability: “subjective probability may rely merely on comparisons of perceived likelihood” (Hawkins & Kapadia, 1984, p.350). In fact, they also refer to the perceptive nature of this kind of thinking. It should be noticed that the Hawkins’ and Kapadia’s (1984) definition of subjective probability does not correspond (but has some connection) to the one provided by de Finetti (1974) as the ratio between the amount of money a person is willing to bet on the outcome of a certain event, and the amount of money he will receive in case of win, if he is willing to change his place with any other one’s place involved in the game. In their paper, Hawkins and Kapadia (1984) related the subjective probability not only to perception but also to coherence. The term ‘coherence’ highlights that any personal judgment should be done in accordance with some ‘axioms’.

Formal probability (ibidem) informs and characterizes the level of theory.

Some features of the ‘a priori’ probability (ibidem) lead to relate it to the arithmetic approach with ratios at the level of discreteness, and the frequentist probability has something to do with the use of percentages. On the one hand, in fact, the assumption of equal likelihood in the sample space that characterize the ‘a priori’ probability may lead to the same misconceptions that arise when the classical definition of probability, as the ratio between successful and possible cases, is used (see also Hawkins and Kapadia, 1984). On the other hand, the frequentist probability is obtained from observed relative frequencies of different outcomes in repeated trials and entails the use of percentages. As the approach at the level of discreteness, the frequentist probability is still linked to the experience, but is an attempt of coming out from the real world alone.

The main differences between the model introduced in this paper and Hawkins’ and Kapadia’s (1984) work are two: the first difference is that the latter does not take into account any semiotic approach, the second one is that the three levels of the former depend on the kind of relations between intuitions and representations, regardless the definitions of probability.

METHODOLOGY

During the academic year 2009/10 at the University of Torino (Italy), data from two undergraduate one-semester 30-hours basic courses in probability were gathered and are under examination. Data consist in written answers to open-ended exercises, which were administered to students all along the semester. 500 undergraduate students were involved: respectively, 150 students in Mathematics (M), and 350 for a master course for Primary school teachers (P). The content of the two courses differed, but in this paper data from a series of exercises that were administered at the beginning of the course are analyzed. Students’ mathematical background of the two
groups differed: while the $P$ students attended one basic mathematics undergraduate course, the $M$ students were attending the second year in mathematics undergraduate course. However, the probability backgrounds were similar in the two groups: both of them did not receive any prior teaching in probability during high school period, or before. This is a very common situation in Italy: unfortunately, probability is taught neither at lower levels of education, nor in secondary schools.

I focus my analysis on the $P$ group, while the $M$ group is referred to for comparisons.

I use an example which helps illustrating the cognitive model and its applications. It is taken from a series of exercises that were given at the beginning of the undergraduate course in probability.

A company employs 75 men (M) and 25 women (F). 12% of M and 20% of F work in the accounts department (a.d.). Compute the probability that, drawing the surname of an employee in the a.d., it would be the surname of a M.

In the exercise, $P(M)$ and $P(F)$, the probabilities of being a M or a F (implicitly: 75 M, or 25 F, over 75+25=100 employees), and the conditional probabilities $P(a.d.|M)$ and $P(a.d.|F)$ of working in the a.d. conditioned to being a M or a F, are given. To solve this problem, it is necessary to compute firstly $P(a.d.)$, and subsequently to compute the conditional probability $P(M|a.d.)$. There is a huge literature concerning the conditional probability in Mathematics Education. For the sake of space, the reader is referred to Huerta & Lonjedo (2007). In their paper, the authors consider three versions (one with percentages, one with probabilities, and one with integer numbers) of a problem, and they show that it is possible to recognize probabilistic or arithmetic thinking when data is expressed in terms of probabilities or percentages respectively. It is well acknowledged that dealing with conditional probability is not immediate for students and needs the use of complex semiotic transformations and counter-intuitive solving strategies. Since the aim of my study is investigating intuitive probability thinking, I consider the case of the a.d. exercise.

Researches focusing on students’ learning processes in terms of intuitions belong to didactics $B$ (didB) that is defined by D’Amore (2006) as an epistemology of learning. According to D’Amore, Didactics $A$ (didA) concentrates on effective teaching environments (assuming implicitly the cognitive transfer). But the couple teaching environments-students does not completely describe the learning process: the teacher, together with his role, his training and his beliefs, constitutes the third component. Researches along the path of the didactics $C$ (didC) (ibidem) focus on the teacher and recently have been spreading, starting from the work by Shulman (1986). It is known, in fact, that for example teachers’ beliefs determine and influence knowledge (didA) and the learning process (didB). Moreover, the relationship between the resources/tools teachers choose and their epistemology belongs to the teacher’s sphere: why did the teacher use that tool? According to which model did he operate such choice? Following this perspective, a more complex research may analyze how teachers’ education and training influence the way they teach and students learn.
(probability) in primary schools. This question, which informs the research presented in this paper, lies at the boundary of didB and didC.

**DATA ANALYSIS**

Table 1 shows the solutions provided by three P students for the a.d. exercise.

<table>
<thead>
<tr>
<th></th>
<th>Simona</th>
<th>Daniele</th>
<th>Emanuela</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td><img src="image" alt="Simona solution" /></td>
<td><img src="image" alt="Daniele solution" /></td>
<td><img src="image" alt="Emanuela solution" /></td>
</tr>
</tbody>
</table>

Table 1: three representations used by P students for solving the a.d. exercise.

Let us look firstly at the representation of Daniele in table 1: he does not solve the exercise, but uses the Venn diagrams to represent the starting point of the exercise: there is the set of M employed in the company, there is the set of employees in the a.d., and the intersection between the two sets provides the number of M in the a.d.. Even if he does not go on with the computations, Daniele is able to come out the experience level and represent the situation in a perceptive and intuitive way, using a pictorial register. Hence, Daniele’s solution belongs to the level of experience. In fact, he does not arrive to a quantitative solution, but nevertheless he is able to give an approximate estimate of a (complex) uncertain situation.

In her solution, Simona operates within the arithmetic register (table 1). No theoretical tool from the probability theory is used, with the exception – in the end of the solution – of the classical definition of probability as the ratio between the number of propitious cases (9M) over the total number of cases (14 employees in the a.d.). This kind of solution has been taken by the majority of P students: they compute firstly the (integer) numbers of M and F working in the a.d., using the percentages and the integer numbers provided by the exercise. Then, working with integer numbers, they sum up the number of M and the number of F, obtaining the total number of employees in the a.d.: 5+9=14. In the end, they compute the ratio
between the number of M in the a.d. and the total number of employees in the a.d. and obtain P(M|a.d.). The solving process is carried out transforming representations within the arithmetic register, and the intuitive schema of proportion is involved. This sophisticated (and correct) solution does not (explicitly) take into account neither the law of alternatives, nor the Bayes theorem. M students do not adopt these solutions, which lie at the discreteness level, and apply the rules of probability. 

M students, in fact, applied the Bayes theorem to solve the exercise, like Emanuela. In the P group, only two students (Emanuela and another one) followed this way. She operates at the level of theory, using a symbolic register and complementing it with a graphic register. The fact that Emanuela uses a graphic register does not imply, however, that she is working at the experience level. Indeed, she is accomplishing a very difficult cognitive task (Duval, 2008), namely coordinating two semiotic registers. It is interesting that Emanuela uses a tree-diagram for representing and computing the probabilities, and in the end she is able to come out the frame of the tree-diagram and apply correctly the Bayes theorem.

The solution of Simona lies at the discreteness level, since she uses ratios. An example of a solution at the discreteness level using percentages is not present in the protocols relative to this exercise. It comes from another exercise:

<table>
<thead>
<tr>
<th>Which is the probability of getting an ace among 52 playing cards?</th>
</tr>
</thead>
</table>

One of the P students, Chifan, does not make computations, but writes only a percentage: 10%. Orally interviewed afterwards, he provides this explanation

Chifan The probability of getting an ace would be around 10%. There are, in fact, 4 aces in the pack of 52 cards. Hence, more or less the probability is 10%.

Chifan does not compute the ratio between the number of successful cases (4) and the total number of cases (52), but assignes a percentage that seemed to him to be likely (and is pretty close to 4/52 = 0.0769). Chifan’s approach does not belong to the experience level only, since he resorts to the arithmetical and percentage registers to represent the situation, but it is not completely into the discreteness level, since approximation and perception are still part of his reasoning process.

As a side remark, this task does not involve conditional probability. The cognitive model presented in this paper, in fact, applies to all uncertain situations, and not only to conditional probability.

**CONCLUDING REMARKS**

In this paper a cognitive model to frame probabilistic reasoning within intuition and semiotic perspectives has been shown. Intuitions and semiotics may have two kinds of relation: they can either support each other, as in the case of Emanuela, who is able to operate complex semiotic transformation (table 1), or impede, as in the case of Daniele, who has strong intuitions, bound to pictorial representations, that do not allow him to come out the experience level and operate proper semiotic
transformation to solve the task (table 1). Supporting or impeding depends both on the immediacy of choosing the representation that is the most proper for the task, and on the transformation(s) the subject operates starting from such representations. Hence, both the individual (together with his abilities, knowledge, etcetera) and the task contribute in determining the supporting or impeding nature of the relation between intuition and semiotic representations.

Future perspectives

Dubinsky’s APOS theory assumes that mathematical knowledge consists in an individual’s tendency to deal with perceived mathematical problem situations by constructing mental actions, processes, and objects and organizing them in schemas to make sense of the situations and solve the problems. The underlying idea of this theory is to extend to the level of collegiate mathematics learning the work of Piaget on reflective abstraction in children’s learning. From a bird’s eye view, it seems that there are many connections between this theory and the model presented in this paper. Hence, it would be interesting to go deeper into details in exploring possible integrations of the two aforementioned theoretical perspectives.

In this paper, only Duval’s semiotic point of view has been taken into account. It would be interesting, in the future, to consider also other semiotic points of view, such as Radford’s Objectification Theory, Arzarello’s APC Space and Godino and Batanero Ontosemiotic Approach.

REFERENCES


