

# MENTAL MODELS OF BASIC STATISTICAL CONCEPTS

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*This report focuses on a research programme that aims to identify the ability to cope with basic statistical concepts of young students (primary school and secondary school) without schooling in statistics, and to identify the students' mental models when acting within simple statistical situations. Firstly, the basic elements of the theoretical framework of the research programme will be outlined. Afterwards the method and exemplary results of a pilot study will be discussed. Finally, these results will be evaluated with regard to follow-up studies.*

Keywords: mental models, basic statistical concepts, statistical knowledge

## INTRODUCTION

About 60 years ago, Piaget and Inhelder (1951) published their seminal work referring to the development of thinking with probabilities. Nowadays, a huge amount of subsequent research exists that, on the one side, replicates the well known levels of the student's development of probabilistic thinking but, on the other hand, modifies, enhances or contradicts the findings of Piaget and Inhelder (Jones, Langrell & Mooney, 2007). Overall, however, we have a substantial knowledge about the development of the (naïve) probabilistic thinking of young students which is a basis to decide on the time to introduce probabilistic concepts in schools (Fischbein, 1975).

In recent years statistics instead of probability seem to become both the essential part of the stochastics curriculum in many countries, and a crucial part of the research into stochastics education (Shaughnessy, 2007). The majority of the existing research approaches in this field are intervention studies, trying to investigate how instruction promotes the students' statistical thinking (Ben-Zvi & Garfield, 2004; Shaughnessy, 2007). In addition, researchers developed theoretical frameworks that highlight crucial aspects, knowledge or competencies towards statistical acting (e.g. Wild & Pfannkuch, 1999). However, in contrast to the research on students' probabilistic thinking, Mokros & Russel (1995) suggested 15 years ago that we knew little about students' development of a statistical thinking without statistical schooling and the situation has not improved today. Nonetheless, empirical knowledge about students' naïve knowledge of basic statistical concepts and statistical situations is important in order to adequately design a statistics curriculum, and to avoid students' misunderstandings referring to statistical concepts or statistical methods (Fischbein, 1975). For this reason in this paper we will discuss the first phase of a research programme that has the main aim:

- to identify the naïve knowledge represented by mental models of young students (aged 9 to 18) regarding basic statistical concepts and situations, and
- to investigate the development of this knowledge depending on age.

To discuss our research approach, firstly, we will clarify basic elements of our theoretical framework. Afterwards, we will discuss the method and provide some examples of the main results of a pilot study of our ongoing research. The pilot study itself consists of the following aims:

- to prove several tasks with regard to their adequateness for investigating students' naïve knowledge towards different basic statistical concepts and situations,
- to get information about whether students actually show different mental models by working with different statistical tasks, and
- to explore whether it is possible to identify differences in the students' ability to cope with basic statistical concepts and situations depending on age.

## **THEORETICAL FRAMEWORK**

The theoretical framework involves two components. The first part deals with the identification of possible subjects and competencies that may represent students' basic statistical knowledge. The second part involves the discussion of students' development of thinking concerning statistical concepts.

### **Statistical Reasoning and Prior Statistical Knowledge**

Wild and Pfannkuch (1999) suggested five aspects to be the main thinking processes in acting statistically. They considered them as being independent from the expertise of the actor. These five processes are

- the recognition of the need for data,
- transnumeration,
- the consideration of variation,
- reasoning with statistical models, and
- integrating the statistical and contextual information, knowledge, conceptions.

Except for the second type of reasoning, these components are not necessarily based on the students' prior statistical knowledge gained through schooling. Thus, it should be possible to identify them as parts of young students' naïve knowledge concerning statistical concepts.

While the five aspects of the statistical thinking represent an individual's statistical acting beyond specific statistical topics, several topics were declared to be central referring to statistical knowledge, i.e. sampling, central measurement and variation, distribution, graphical representation of data, or regression and correlation (e.g. Curcio, 1989; Mokros & Russel, 1995, Bakker 2007).

## **Students Development Concerning Statistical Thinking**

A main assumption for this research is that students' competencies in basic statistical concepts develop with age. Unlike Piaget and Neo-Piagetians like Case (1992) we assume that this development cannot be described adequately by a staircase model. We follow Siegler (1996) who postulated the development of children's thinking had to be considered as "overlapping waves". The fundamental idea of his approach is making allowance for the observable variability of children's thinking. This variability depends on different factors, like specific circumstances, requirements and available knowledge, which influence a child's concrete actions within different situations. Based on several empirical studies Siegler provided evidence for high inter-individual as well as intra-individual variability when using complex strategies for solving problems (e.g. Siegler, 1995; Schauble, 1990).

The theory of mental models (Johnson-Laird, 1983) incorporates the situation's impact on cognitive processes, too. This theory suggests that, when interacting with (statistical) demands of a specific situation, the learner builds a mental model in order to simulate relevant aspects of the situation to be cognitively mastered (cf. Seel, 2001). Mental models are of dynamic nature: They are not to be seen as being fixed structures of memory; mental models are constructed according to a task and its requirements within a situation representing the structure or the function of the modelled object (Schnotz & Bannert, 1999). Thus, mental models concern the situation and they also facilitate to differentiate the students' cognitive development (or maturation) in using basic statistical concepts. Furthermore, tasks concerning statistical thinking and statistical knowledge contain information a student could proceed and internalize in a mental model being specific with regard to his or hers individual abilities, pre-knowledge and apperception of the task's representation. According to the information process model of Schnotz & Bannert (1999), the elements of a mental model may be changed, enriched or modified during the persistent mutual processes of internalizing and externalizing, when a student is working on a statistical task but they do not disappear at all. Thus, information about statistical mental models, and hence statistical thinking and statistical knowledge, could be made available by analyzing the tasks (content, representation), the learners specific situation (experience and pre-knowledge with regard to statistical content, statistical methods and statistical context) and the learners outcomes (written or spoken responses) after working on the tasks.

Not the statistical mental models themselves are claimed to be captured, but the researchers' reconstruction of these models based on the observed and recorded learning outcomes. It is obvious that different students will show different grades of performance. A categorization system describing different performances that is often used in statistics education is based on the SOLO model of Biggs and Collis (1982). In the adaptation of Watson and Moritz (2003), this model has four levels or rather modes, i.e.

- the prestructural mode: students solve a task using irrelevant information,
- the unistructural mode: students solve a task using an isolated information,
- the multistructural mode: students solve a task using a set of information,
- the relational mode: students solve a task using a set of information and considering an interconnected knowledge of context and statistical concepts.

Although alternative models exist to describe students' performances in acting with specific statistical tasks (e.g. Watson et al., 2003), in the pilot study we used the model outlined above only, but consistently, since the scope of statistical tasks in the pilot study encompassed different basic statistical concepts.

In summary, using this theoretical framework the underlying aim of the pilot study was to get information about students' ability to cope with basic statistical concepts within different situational requirements. A further aim was to see if and how it could be done to reconstruct students' potential mental models when working on different statistical situations.

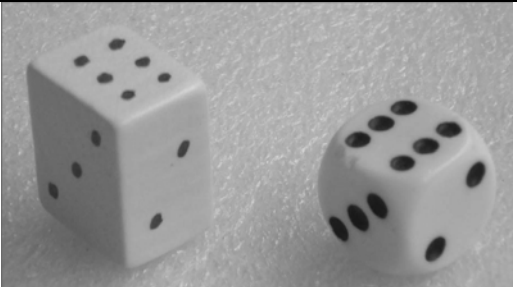
## METHOD

To investigate different aspects of students' naïve knowledge concerning basic statistical concepts and situations we have designed tasks referring to central measurement, variability of statistical data, proportional reasoning involving the students consideration of variability, interrelation of bivariate data, and simple random experiments.

The design of the tasks was based on the research literature as mentioned above. Some of these tasks were adopted without changes, some other tasks were modified, and some new tasks were developed. Every task contains a decision-making process in a statistical situation and an open-ended item in which the student had to justify the decision. The particular aim of the pilot study was to investigate whether the students of different age were able to work with these tasks and whether these tasks could be useful to identify students' mental models when acting with different statistical situations.

Figure 1 shows two of the tasks. The first task, called the *frog-task*, deals with the students' mental models referring to proportional reasoning and variability of statistical data. The second task, called *the die-task*, deals with the students' mental models referring to a simple random experiment (Figure 1).

Both items have adequate normative solutions. For instance, estimating 100 frogs in field 3 after 100 jumps or choosing the ordinary die in the second task are inadequate solutions. Our main focus in analysing students' solutions, however, was to code students' different performances in justifying their solution according to the four-level-model of Watson and Moritz (2003).

<p style="text-align: center;">Field 1   Field 2   Field 3   Field 4</p> <div style="text-align: center; margin-bottom: 10px;">  </div> <p>Andrea lets the frog jump ten times. The frog ends in field 1 once, in field 2 twice, in field 3 seven times, and never in field 4.</p> <p>Make an estimation of results of the following experiments:</p> <p>How often will the frog get to field 3 after 100 jumps? Justify your answer.</p> <p>How often will the frog get to in field 3 after 1000 jumps? Justify your answer.</p> <p>How often will the frog get to field 4 after 1000 jumps? Justify your answer.</p>	<div style="text-align: center; margin-bottom: 10px;">  </div> <p>You will win a game if one die shows 4 the first time you throw. Which of the two dice would you choose to win this game?</p> <p>Why have you chosen this die!</p>
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**Figure 1: The frog-task (left side) and the die-task (right side)**

In this study we coded no solution with 0, a solution that matches the prestructural mode with 1, a solution that matches the unistructural mode with 2, a solution that matches the multistructural mode with 3, and, finally, a solution that matches the relational mode with 4. We also tried to identify different mental models that were, potentially, the basis of the students' justifications of their solutions.

die-task		frog-task	
Grade [age]	Sample size	Grade [age]	Sample size
2 [7-8]	19	4 [9-10]	21
4 [9-10]	25	10 [15-16]	45
8 [13-14]	65	12 [17-18]	23
11 [16-17]	17		
n = 126		n = 89	

**Table 1: Sample sizes in the pilot study**

To identify possible differences in the performances of the students' solutions depending on both the students' age and the students' specific situation (experience and pre-knowledge with regard to statistical content, statistical methods and statistical context; see above), we selected samples of students in different grades. The samples answering the two tasks we discussed above are shown in Table 1.

All primary students and most of the students in higher grades had never learned about statistics in school before taking the test. There were only a few students who had gained mostly little statistical experience along their schooling.

As we proceed, we begin with the discussion of students' solutions of the die-task. Afterwards, we will discuss the results in a broader sense including open questions that have arisen from the results of our first research step, and including possible next research steps.

## **SOME RESULTS OF THE DIE-TASK**

The die-task provides the following question (see Figure 1): "Why have you chosen this die?"

A grade 4 student who chose the ordinary die justified his solution as follows: "Because, it will spin better". This student provided an inadequate solution on the one side and, on the other side, referred to a physical feature of the die, which is irrelevant in the statistical situation of choosing one of the two dice. For this reason, we coded his solution with 1 (representing the prestructural mode).

Another grade 4 student justified his selection of the cubic-die as follows: "Because the side is bigger. Thus, you get the four faster." The student referred to a single physical feature of the die which is relevant for an adequate choice of one of the two dice. The student was not able, however, to describe the relationship between the different sizes of the sides of the cuboid-die, and, respectively, the different sizes of the side showing the 4 on both dice. For this reason, we coded this justification with 2 (unistructural mode).

A third student (grade 8) justified his selection of the cuboid-die as follows: "Because the first die has sides of the same size, which is not the case for the second one". This student compared the two dice according to a relevant physical feature, and, thus, used a set of relevant information in the statistical situation. For this reason, we coded this justification with 3 (multistructural mode).

Finally, one student (grade 8) justified her selection of the cuboid-die as follows: "The sides showing the 3 and the 4 are bigger than the others. For this reason, the chance of getting a 4 is about  $\frac{2}{8}$  (two eighth), thus  $\frac{1}{4}$  (one fourth). This chance is bigger than  $\frac{1}{6}$  (one sixth) in the other die." This student considered a lot of relevant information. She compared the symmetrical areas of the cuboid-die showing the 3 and the 4 and estimated a probability by comparing all the areas of the cuboid-die. She also showed knowledge about the probability of the ordinary die and was able to compare these probabilities. For this reason, we coded this justification with 4 (relational mode).

In Table 2 the absolute numbers of students' justifications in the whole sample (n=126) that were coded from 0 to 4 is shown. The justifications are divided into those that are given concerning the cuboid-die (r: right) and the ordinary die (w: wrong).

Code	Grade 2 n = 19	Grade 4 n = 25	Grade 8 n = 65	Grade 11 n = 17
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	r	w	r	w	r	w	r	w
0	0	0	0	0	0	0	0	0
1	5	14	4	12	8	8	1	1
2	0	0	6	0	44	0	12	2
3	0	0	2	1	4	0	1	0
4	0	0	0	0	1	0	0	0
Sum	5	14	12	13	57	8	14	3

**Table 2: Absolute numbers of codes concerning the students' justifications**

Three results of the descriptive analysis can be pointed out: First, the students' ability to select the adequate die, i.e. the cuboid die, seems to increase by age. Second, the students' performance in formulating a justification for their choice of one of the two dice seems to increase by age, as well. And third, if the students choose the inadequate die, they show a low performance in justifying their choice.

### SOME RESULTS OF THE FROG-TASK

The coding of the students' justifications in the frog-task was based on considerations similar to the die-task. Although we coded the justifications to the four answers of the frog-task as a whole, we illustrate the codes from 1 to 4 by presenting only justifications to the last question ("How often will the frog get to field 4 in 1000 jumps?"):

- 1: "250 times. The chance that the frog will get to each of the four fields is 25%."
- 2: "0 times. 1000 is a hundred times larger than 10 and, thus, we have to multiply 0 (number of frogs having reached field 4 by jumping 10 times) by 100."
- 3: "8 times. Because getting to field 4 seems to be more difficult."
- 4: "0-10 times. The more often the frog will jump, the bigger is the chance that the frog will get to field 4."

The justification of the first student (grade 10) describes an equal-probability model, which is an inadequate model in the statistical situation of the frog-task. The justification of the second student (grade 12) describes a fixed proportional model without consideration of variation. In contrast, the third student (grade 4) seems to grasp the idea of variation implicitly and in an unsophisticated way. Finally, the fourth student (grade 12) showed a more sophisticated understanding of variation and of the relationship between variation and the number of jumps. Table 3 shows the absolute numbers of students' justifications in the whole sample ( $n = 89$ ) that were coded from 0 to 4.

Code	Grade 4 n = 21	Grade 10 n = 45	Grade 12 n = 23
0	0	1	0
1	7	5	3

2	10	19	11
3	4	17	6
4	0	3	3

**Table 3: Absolute numbers of codes concerning the students' justifications**

Regarding the results of the descriptive analysis, it is important to state that the performance of the students' justifications seems to increase with the student's age. However the increase is low in comparison to the die-task and, for instance, one student of grade 4 (see above) showed a higher performance as a student of grade 10 and even a student of grade 12. Furthermore, the performance of the students' justifications is higher compared to the justifications referring to the die-task.

## DISCUSSION

Being the first step of our on-going research the results of this pilot study have to be interpreted carefully. These results, though, should facilitate the finding of appropriate questions and the phrasing of adequate hypotheses referring to students' ability to cope with basic statistical concepts. Taking into account these constraints the following aspects seem, nevertheless, to be worthwhile being considered in follow-up studies.

The results concerning the two tasks as well as for other tasks in our tests gave evidence that students' ability to cope with basic statistical concepts and to justify decisions in different statistical situations are dependent on age (see the results in both tasks). This hypothesis will be investigated in a follow-up study involving randomised samples of students of different grades. Although there seems to be a correlation between the students' performance concerning different tasks, students of the same age also justify their acting in a statistical situation in considerably different ways (see for example the two students of grade 12 concerning the frog-task). Furthermore, there is some evidence that, partially, students formulate different justifications in tasks that are structurally equivalent. Both results meet the Siegler's theory of "overlapping waves" in children's thinking (Siegler, 1995). There is also some evidence that students use different mental models coping with statistical situations. Although it is, theoretically, not possible to identify the mental model a student actually uses, research results seem to reveal some indication of equivalent inter-individual mental models. For instance, some of the students neglected in both tasks the randomness of the data's genesis or, respectively, the randomness of future data (e.g., the student we coded with 0 in the die-task). Another pattern concerns the omnipresence of the fair chance: Neglecting the information given in the frog-task, these students estimate future events consistently based on the model of an equal-probability (see the student, we coded with 0 in the frog-task).

According to the theory of mental models, we found great differences in the students' ability to cope with statistical situations depending on the representation of the task. For instance, students' seemed to use significantly different mental models when the

die-task was represented by two real dice, the picture of the dice (c.f. the die-task discussed above), or the description of the two dice. We also hypothesise that different representations of other statistical situations will have a great impact on the students' ability to cope with these statistical situations. For instance, the frog-task using the picture of the statistical situation shown in Figure 1 could be compared to the same task involving a picture of a real experiment or more abstract graphical representations of the situation, e.g. a dot plot.

Two important questions were not investigated in our pilot study: Do we measure the students' ability to cope with statistical situations or do we only measure the students' ability to communicate their ideas? Do the differences in communicating decisions and their justifications in a statistical situation contrast with a student's acting in the same situation? The outward appearance of the written justifications of students of different age varies considerably (c.f. Figure 2 to 5). At the current stage of our research programme, however, we postulate that a student's ability to cope with statistical situations is similar to the student's ability to verbalise his decisions and his decisions' justifications in a statistical situation. Nevertheless we will subsequently investigate qualitatively whether there are differences between students' verbalisation of decision-making in statistical situations and students' acting in these situations. Two follow-up-studies are being undertaken currently on the dependence of students' ability to cope with basic statistical concepts, on the students' different mental models by coping with statistical situations and on the students' ability to cope with basic statistical concepts in dependence of the situation's representation. From both studies which will include quantitative and qualitative parts we expect to get a better understanding of students' (naïve) knowledge concerning statistical concepts and students' (naïve) statistical thinking.

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