EXPLORING THE SOLVING PROCESS OF GROUPS SOLVING REALISTIC FERMI PROBLEM FROM THE PERSPECTIVE OF THE ANTHROPOLGICAL THEORY OF DIDATCITS

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Abstract: This paper reports on a first attempt to use the notions of ‘Research and Study Course’ (RSC) and ‘praxeologies’ within the Anthropological Theory of the Didactics (ATD) to analyse groups of students engaged in the mathematical activity of solving realistic Fermi problems. By considering so called realistic Fermi problem as a generating question in a RSC the groups’ derived sub-questions are identified and the praxeologies developed to address these are discussed.

INTRODUCTION AND BACKGROUND

Working with mathematical models and modelling is a central part of the national intended written curriculum for the Swedish upper secondary mathematics education (Skolverket, 2000). Indeed, although without any definitions provided and only implicitly described, the notions of mathematical modelling and models have successively been explicitly more emphasised in the last two curricula reforms from 1994 and 2000 respectively (Ärlebäck, 2009a). Generally however, mathematics education research involving mathematical models and modelling at the Swedish upper secondary level is largely under-researched, and interest in research with this particular focus is just in its infancy. Nevertheless, one of the most palpable conclusions from this initial research is the big discrepancy found between the indented written curriculum and what the students actually attain (Ärlebäck, 2009b). For example, in a study of 381 third year upper secondary students 77 % stated that they never had encountered the notions during their upper secondary education (Frejd & Ärlebäck, accepted). Part of the problematique might be that teachers have difficulties in formulating and explaining their conceptions of these notions (Ärlebäck, 2010), or that mathematics teaching at this educational level in Sweden strongly is influenced by ‘traditional’ textbooks (Skolverket, 2003) with little or no discussions about models and modelling. However, it has been suggested and concluded that the introduction of, and students’ initial conceptualisation of, mathematical modelling (interpreted in line with the written curriculum documents) at the upper secondary level adequately and efficiently can be done using so called realistic Fermi problems (Ärlebäck, 2009c; Ärlebäck & Bergsten, 2010). The aim of this paper is to continue, deepen and extend this line of investigation.

MATHEMATICAL MODELLING AND REALISTIC FERMI PROBLEMS

In the research literature in mathematics education there are many different perspectives on and ways to approach mathematical modelling (e.g. Blum, Galbraith,
Henn, & Niss, 2007; Lesh, Galbraith, Haines, & Hurford, 2010). Concepts and notations used are for instance those of *competencies* (Blomhøj & Højgaard Jensen, 2007; Maaß, 2006); *modelling skills* (Berry, 2002); and, *sub-processes or sub-activities* (Blomhøj & Højgaard Jensen, 2003). These all focus on the descriptions of, relations between and/or the transitions of phenomena in the real world and their mathematical representations. From a Swedish perspective, the intended written curriculum (e.g. Skolverket, 2000) permits a broad interpretation of the meaning and content of the notions of mathematical models and modelling (Årlebäck, 2009b). One such interpretation with a real influence on the school practice has been presented by Palm et al. (2004), which is used for the construction of national test items focusing on mathematical modelling [1]. In principle, the interpretation by Palm et al. concord with the view on modelling illustrated in Figure 1.

![Diagram](image-url)

**Figure 1:** The modelling cycle as presented by Borromeo Ferri (2006, p. 92) after adaption from Blum and Leiβ (2007)

Based on this view on modelling Ärlebäck (2009c) and Ärlebäck and Bergsten (2010) explored the idea that so called realistic Fermi problems are ‘modelling problems in miniature’, which potentially could be useful and time effective for introducing some of the typical features of mathematical modelling; see Ärlebäck (2009c) for details. Realistic Fermi problems are characterized by (1) their *accessibility*, meaning that they can be approached by all individual students or groups of students as well as be solved on both different educational levels and on different levels of complexity. Normally, any specific pre-mathematical knowledge is not required to provide an answer; (2) their clear real-world connection, to be *realistic*; (3) the need to *specify and structure the relevant information and relationships* to be able to tackle the problem. In other words for the problem formulation to be open in such a way that the problem solvers not immediately associated the problem with a know strategy or procedure on how to solve it, but
rather urge the problem solvers to invoke prior experiences, conceptions, constructs, strategies and other cognitive skills in approaching the problem; (4) the absence of numerical data, that is the need to make reasonable estimates of relevant quantities; and (5) their inner momentum to promote discussion, that as a group activity they invite to discussion on different matters such as what is relevant for the problem and how to estimate physical entities (e.g. respectively (3) and (4) above) [2].

The realistic Fermi problem the groups of students solved used in Ärlebäck (2009c) and Ärlebäck and Bergsten (2010) was the Empire State Building problem:

**The Empire State Building problem (ESB-problem):**

On the street level in Empire State Building there is an information desk. The two most frequently asked questions to the staff are:

- How long does the tourist elevator take to the top floor observatory?
- If one instead decides to walk the stairs, how long does this take?

Your task is to write short answers to these questions, including the assumptions on which you base your reasoning, to give to the staff at the information desk.

The data from three groups of students working on the ESB-problem using only paper and pencils for approximately 30 minutes was analysed using a developed analytical tool called Modelling Activity Diagram (the MAD framework). This framework is inspired by Schoenfeld’s ‘graphs of problem solving’ (Schoenfeld, 1985), the view adapted on mathematical modelling described above, and the five characteristic features of realistic Fermi problems. It pictures the different types of sub-activities the groups engage in during the problem solving process in terms of the categories **Reading**, **Making model**, **Estimating**, **Verifying**, **Calculating**, and **Writing** (see Figure 2 and Ärlebäck (2009c) for details). However, although this macroscopic schematic representation clearly shows that the sub-activities are dynamically and richly represented in the solving process of the Fermi problem (cf. Figure 2), it does not provide any detailed information about what kind of discussions, topics and questions the groups addressed and investigated. In order to get a more nuanced and circumstantial picture of the problem solving process involving realistic Fermi problems in these respects this paper aims to provide a more microscopic analysis focusing on what actually is discussed within the groups, especially in connection to mathematical topics and content. Hence, the research question studied in this paper can be states as follows: What are the questions the students address during the problem solving process of the EBS-problem? What mathematics do the students use in the problem solving process and how is the use of this mathematics motivated and justified?
Figure 2: An example of a Modelling Activity Diagram of one of the groups solving the ESB-problem (Ärlebäck, 2009c, s. 346).

THEORETICAL FRAMEWORK, METHODOLOGY AND METHOD

This paper uses the notions of praxeologies and Research and Study Course (RSC) from ATD. Within this framework praxeologies are used to describe any human activity in terms of two ‘blocks’: a praxis block (‘know-how’ or ‘practical-part’) containing both a designated type of tasks and the techniques used/needed to complete/perform these; and a logos block (‘know-why’ or ‘knowledge-part’) containing the technologies that explain, justify and describe the techniques as well as the formal justification of these technologies, the theory. As the name praxeologies suggests the praxis- and logos blocks are to be regarded as inseparable (Barbé, Bosch, Espinoza, & Gascón, 2005; Rodríguez, Bosch, & Gascón, 2008).

The notion of Research and Study Course (RSC) introduced by Chevallard (2004; 2006) is a general model that can be used for both designing and analyzing learning and study processes. A main emphasis of a RSC is on the generating question, \( Q_0 \), which should be intriguing and of genuine interest to the students as well as ‘rich enough’ to encourage the students to derive, pursue and answer dynamically raised and related (sub-)questions, \( Q_i \), in the quest of trying to arrive at an answer to the question \( Q_0 \). In addressing these questions the students have to invoke, use and/or develop one or more praxeologies. The derived sequence of sub-questions \( Q_i \) and their respective answers \( R_i \) are often represented and illustrated in a ‘tree-diagram’ which illustrates the relationships between the different studied questions \( Q_i \); see Figure 3 for an example.
In terms of ATD the study reported on in Ärlebäck (2009c) and Ärlebäck and Bergsten (2010) can be conceptualised as an investigation of the didactical praxeology with the task to introduce mathematical modelling to students at the upper secondary level using the suggested technique presented by realistic Fermi problems. The issues addressed in these papers, as well as in this one, are concerning the (underdeveloped) logos block of this praxeology, especially the technology part. The ATD concepts of RSC and praxeologies provide theoretical constructs focusing both on what questions the students tackle when solving the ESB-problem as well as how and why. In the notion briefly introduced above the research question(s) studied in this paper can be reformulated as: Given the ESB-problem as a generating question in a RSC, what sub-questions are addressed by the participating groups of students and what (mathematical) praxeologies are used and developed? Due to space limitation the main emphasis in this paper will be on the first of these questions.

From an ATD perspective García et al. (2006) have presented a conceptualisation of mathematical modelling which basically equates all mathematical activity with mathematical modelling. In this paper however, the view of modelling is inherited from Ärlebäck (2009c) as briefly described and argued for in the previous paragraph.

To address the research question, widening and deepening the analysis of the groups of students solving realistic Fermi problems, the recorded video and transcribed data from one of the groups used in Ärlebäck (2009c) was revisited and re-analysed. The basic idea was to consider the students’ work on the realistic Fermi problem in the context of a SRC as the generative question $Q_0$ and to see what (sub-)questions $Q_{i,j,...}$ this led the students to investigate, and in addition to link these questions to the MAD representation of the problem solving process of the studied group. Note that in the ESB-problem the generating question, $Q_0$, actually is two questions:

$$Q_{0,1}: \text{ How long does the tourist elevator take to the top floor observatory?}$$

$$Q_{0,2}: \text{ If one instead decides to walk the stairs, how long does this take?}$$

The approach taken was in line with Hansen and Winsløw (2010) who make use of the RSC as an analytic model. Focus of the analysis was on the group activity as whole and thus firstly on explicit questions uttered by the members of the group, and secondly on how these questions were addressed in terms the constituents of one or more praxeologies. Although there exist an a priori analysis in Ärlebäck (2009c) identifying some of the questions the problem solvers need to address in order to solve the problem, this paper only focus on the empirical questions actually addressed by one of the groups of students during their problem solving session.

RESULTS

The questions $Q_{i,j,...}$ the students derived from the generative questions and examined are presented below in the order in which they were raised and posed during the
problem solving session. The formulations below are in principle the students’ own wording; some minor alterations have been made in order make the actual question intelligible and more concise. Basically the questions \( Q_1 \) are concerned with the ESB’s physical appearance, \( Q_2 \) address \( Q_{0,1} \) (taking the elevator), and \( Q_3 \) address \( Q_{0,2} \) (taking the stairs):

\( Q_1 \): How high is the Empire State building?

\( Q_{1,1} \): How many floors are there in the Empire State Building?

\( Q_{1,1,1} \): How high is a floor?

\( Q_{1,2} \): How high can a general building be?

\( Q_{1,3} \): How high was the World Trade Centre?

\( Q_2 \): How fast is an elevator?

\( Q_{2,1} \): What is the weight of the elevator?

\( Q_{2,1,1} \): How much work is being done by the elevator?

\( Q_{2,1,1,1} \): Given the work done by the elevator, can we then calculate its velocity?

\( Q_{2,2} \): How long does it take to ride the elevator to Michael’s apartment [a friend]?

\( Q_{2,2,1} \): On what floor is Michael’s apartment?

\( Q_3 \): How tired does one get from walking the stairs?

\( Q_{3,1} \): How longer does it take for one floor?

\( Q_{3,2} \): How much longer does it take for each floor?

\( Q_{3,2,1} \): How long does it take to walk up the first floor?

\( Q_{3,2,1,1} \): How fast is normal walking pace?

\( Q_{3,2,1,2} \): What is the inclination of the stairs?

\( Q_{3,3} \): My [one of the students] mother lives on the fifth floor – I wonder how long it takes walking up the stairs to her place?

The relationships between these (sub-)questions are illustrated in Figure 3. Note that the dotted lines in the tree-diagram display the dependence of the answers \( R_1 \), \( R_2 \), and \( R_3 \) respectively with respect to previously answers to questions in the tree. However, due to space limitations, and the fact that the focus of this paper is on the derived questions, these details are omitted here to be discussed elsewhere.

All branches except \( Q_{2,1} \) represent questions which answers contributed to the solving of the ESB-problem. The branch \( Q_{2,1} \) is about the classical mechanics concept of work, which the students briefly discuss as one possible strategy to get an estimate for the velocity of the elevators in the ESB.

After having spent about 15 minutes on the problem the group starts working on details concerning their suggested model on how to take the physical exertion into consideration in the \( Q_{0,2} \) question. They continue to do this in approximately 4 minutes, before the writing of the letter instructed in the problem formulation begins.
CONCLUSION AND DISCUSSION

One can notice that the actual modelling in terms of discussing, structuring and determining central variables and relationships important for solving the problem is something that is made implicitly and silently throughout the problem solving session. The praxeologies developed to address the questions (tasks) $Q_{0,1}$ and $Q_{0,2}$, in fact all three groups in Ärlebäck (2009c) used the mathematical model $t = s/v$ ($t$ being the time, $s$ the distance, and $v$ the (average) velocity) as the basic technique to approach the questions. However, the decision to use this model was not explicitly uttered, or in any other way directly communicated, within the groups; it seems that
all the participating students took it for granted that this was the model to use to solve the problem. In other words, the logos of this praxeology is kept hidden. It is possible that this ‘choice’ narrowed the groups’ possibilities to go beyond this model and come up with more elaborated models.

Figure 4: The Modelling Activity Diagram (Ärlebäck, 2009c, s. 346) extended with the order and dynamics of the derived questions in the RSC.

A majority of the praxeologies the students developed made use of estimation as the technique to resolve the tasks originating from all (but \(Q_{2,1,1,1}\) and \(Q_3\)) of the derived questions the students engaged in. All these praxeologies have underdeveloped logos and the technologies and theories invoked to justify and verify the estimates are based on personal and often anecdotal experiences. This is most probable due to the feature of realistic Fermi problem to not provide the students with any explicit numbers to work with. It should be noted that one of the technologies applied and made use of to validate the estimate in some of these praxeologies are the same mathematical model as used as the technique in addressing \(Q_{0,1}\) and \(Q_{0,2}\); \(v=s/t\).

The result suggests that there are some often used mathematical models, here exemplified by \(v=s/t\), which are taken for granted used without second thought and reflection on underlying assumptions, limitations or alternatives. An explanation might be found in the different institutional conditions and constrains where these models are taught, learnt, practiced and applied. In particular, it would be interesting
to study the didactical transposition of the notions and use of mathematical models and modelling to see where these conditions and constrains arise.

Though it has proven productive and useful to use realistic Fermi problems for the introduction of mathematical modelling at the upper secondary level, the challenge for the future is to design generative questions in the RSC so that also more advanced mathematical praxeologies are invoked and developed. The RSC ‘allows’ for the teacher to intervene, comment and make suggestions during the course of study, and this present a possibility achieve more, and perhaps specific, advanced mathematical content

NOTES

1. For a critical discussion of this interpretation and its possible consequences see Ärlebäck (2009b).
2. For a discussion on the relations between realistic Fermi problems and other characterisations of problems (such as modeling eliciting activities, numerosity problems etc.) see Ärlebäck (2009c).

REFERENCES


