MATHEMATICAL CHALLENGING TASKS IN ELEMENTARY GRADES
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Mathematics teaching and learning should include much more than routine tasks, which appeal to memory and drill, being complemented by others, mathematically more rich and challenging, as problem solving and investigations, that push for reasoning, creativity and connections. So teachers must have the ability to choose, formulate and adjust tasks according to their intended objective. In this paper we present some mathematical pattern tasks to use in elementary mathematics classes that, intending to be challenging, can develop students higher order thinking and creativity.

INTRODUCTION
The new Portuguese curriculum proposes a different perspective about the learning and teaching of school mathematics, with great challenges for both teachers and students. Despite of being well written and transparent about its aims, pathways for implementing these changes in real classroom settings are not fully defined and can be very complex and difficult to accomplish by the teachers when they haven’t innovative instructional materials, and in fact only a few are available yet. More then ever we only can have students engaged in knowledge and critical citizenship if school and teachers promote creativity in their own classroom - being creative both in developing curriculum to get higher levels of students understandings through adequate tasks and in promoting creativity in students work. In this setting we are designing a two-year project to develop a research-based professional development curriculum focused on mathematics tasks, which will assist teachers in their practices and teachers in their initial preparation of grades k-6. Our attention is directed not only to mathematical themes of the curriculum but also to cross-mathematics processes - communication, problem solving and reasoning. Thus we intend to draw sequences of tasks, materials, expectations for each, and methodological notes about their use, and in particular selecting illustrative episodes of students’ creative resolutions of the tasks and teachers’ creative ways of exploring them.

THEORETICAL FRAMEWORK
Teacher education and school mathematics

Literature identifies the major obstacles to reforms as teachers’ lack of familiarity with innovative instructional practices and tools, teachers’ lack of understanding of mathematics they teach and their inability to communicate mathematics with students in ways other than direct instruction; and teachers’ reluctance to conform to new
methods of teaching due to their beliefs about what students need to know (Heibert et al., 2007).

Several researchers (Raymond, 1997; Remillard & Bryans, 2004; Schoenfeld, 2008) suggest that teachers’ knowledge, beliefs and attitudes influence their actions in the classroom and their interactions with students. Towards a mathematics preparation, Ma (1999) claims that teachers must have a profound understanding of fundamental mathematics to provide teaching and learning processes. In this pathway reflection plays an important role as the reconstruction of the teachers’ experience and knowledge (Hodgen, 2003). Actually, teachers with more explicit and organized knowledge tend to use with their students more conceptual connections, appropriate representations and active student discourse (Warren, 2006).

Nowadays, widespread tendencies in mathematical education suggest that effective learning requires that students be active and reflexive when they are involved in significant and diversified activities. This idea follows a way of thinking where higher order and critical thinking skills are privileged, where lectures are substituted by dialogue and discovery methods. According to Boaler (2002), different teaching methods are not just vehicles to produce more or less knowledge, they shape the nature of knowledge production and define the identity of students toward mathematics through the practices in which they involve. As such, teachers must have an in-depth understanding of the mathematical thinking of their students. In doing so, they can support the development of their mathematical competence (Franke et al., 2007). Further, they need an understanding of how to mobilize this knowledge for their students’ learning. We believe that students’ mathematical thinking must support teachers’ practice so teachers must construct or adapt good mathematical tasks to use in the classroom. They should have the capacity to be creative for themselves in the tasks they propose but be also mathematically competent to analyse their students’ resolutions.

Research shows that learning heavily depends on teachers. They must make a set of decisions during the instructional process that depend on various factors affecting their actions, including how to interpret the curriculum and select curricular materials and strategies to use in the classroom. Within this context we as teacher educators have to design good mathematical tasks that are used to achieve a variety of goals, in particular further mathematical understanding and creative thinking in order to motivate students to learn. So we have the dual responsibility of preparing mathematics teachers, both mathematically and didactically, and discuss the way tasks can be designed and refined for the purposes of promoting mathematical understanding.

**Mathematics tasks and the development of mathematical knowledge**

Having an explicit understanding of how and why mathematical tasks are used in teacher education gives us insight into what qualities a good task must have. However, it does not automatically provide us with the ability to design good tasks.
The designing of good tasks requires an interface between the theoretical and the practical, between the intended and the actual, between the task and the student. The process of designing mathematical tasks is a recursive one that applies to the creation of entirely new tasks as well as it does to the adaptation (or refinement) of already existing tasks (Liljedahl et al., 2007). Serpinska (2003) regards the design, analysis and empirical testing of mathematical tasks, whether for the purposes of research or teaching, as one of the most important responsibilities of mathematics education.

What students learn is largely influenced by the tasks given to them (Stein & Smith, 1998). In fact, the tasks used in the classroom provide the starting point for the mathematical activity of students (Doyle, 1988) and the way of implementing a task determines its cognitive level. Its nature significantly influences the type of work that is done in math class; they should be diverse in nature and in context, giving rise to a variety of representations, using different resources and promoting discussion. Discussion that focuses on cognitively challenging mathematical tasks, namely those promoting flexible thinking, reasoning and problem solving, is a primary mechanism for promoting conceptual understanding of mathematics (Smith et al., 2009). Such discussions give students opportunities to share ideas and clarify understandings, develop convincing arguments, develop a language for expressing mathematical ideas and learn to see things from other perspectives (NCTM, 2000).

Although discussions about higher-level tasks provide important opportunities for students to learn and to promote creativity in their resolutions, they also present challenges to the teacher who must determine how to organize discussion built from a diverse set of responses. The teacher must decide what aspects of a task to highlight, how to organize the work of students, what questions to ask to challenge those with different levels of expertise and how to support students without taking over the process of thinking for them and thus eliminating the challenge (NCTM, 2000). Giving students too much or too little support or too much direction can result in a decline in the cognition demands of the tasks (Stein et al., 1998). On the other hand, mathematical challenging tasks are not just difficult tasks or having a higher level of mathematization (Holton et al., 2009; Stillman et al., 2009) and much of the challenge may be provided by the teacher.

Patterning tasks are a specific kind that allows a depth and variety of connections with all topics of mathematics leading to consider patterns as cutting across all of mathematics education, both to prepare students for further learning and to develop skills of problem solving and communication (NCTM, 2000; Orton, 1999; Polya, 1945; Vale et al., 2009). Thus we will give special attention to this kind of tasks, mainly for representations they raise (very different and usually taking the shape of analogies, drawings, manipulative or tables). Research shows that the use of multiple representations is beneficial in the teaching and learning of mathematics (Tripathi, 2008). Our previous research analyzed the impact of an intervention centered on the study of patterns in the learning of mathematics concepts (Vale et al., 2009; Barbosa, 2009). Since patterns are powerful in the mathematics classroom and can suggest
numerical, visual and mixed approaches (Orton, 1999) and exploring growing patterns in elementary levels lays in the foundation for the algebraic reasoning (e.g. Usiskin, 1999, Rivera & Becker, 2008) we designed a didactical experience grounded on figural patterns as a suitable context to get expression of generalization and contribute to approach algebraic thinking.

The research has also shown that pattern tasks are a fruitful focus to support teacher inquiry and students learning towards the implementation of the current curriculum orientations. Moreover, patterning tasks challenge for different representations and to look for creative ways to reach the solutions; for instance, if students look for different ways of counting a collection of elements in figures, or a general rule of a growing figurative pattern. We identify the five different interconnected representations proposed by Clement (2004) for the mathematical ideas: pictures; manipulative, written symbols, spoken language and relevant situations. They must be used in mathematics lessons in order to provide a lens for making sense of students’ resolutions and responses and can be a guide for teachers to plan their lessons. We privilege problem tasks that require manipulative because, in doing so, specially young students seem to create a more significant and long-lasting experience, becoming involved in their own learning (NCTM, 2000; Weiss, 2006; Vale, 2003).

**Challenge and creativity in the mathematics classrooms**

We can read in ICMI Study 16 (Barbeau & Taylor, 2005) that mathematics is engaging, useful, and creative. The sentence itself was a challenge for us in the way that it conducted us to wonder what can we do to make it accessible to our students. In the last decades we developed our work and research around problem solving which led us to believe that it can be a fruitful context to engage both students and teachers to perceive those mathematics characteristics. A problem solving approach can reflect the creative nature of mathematics and give students opportunities both to learn mathematics and to feel the way in which mathematicians develop mathematics.

Learning mathematics is much more than facts, memorizing and mastering rules, techniques and computational algorithms, despite their importance and role. It entails incorporating experiences and conceptual understanding to solve different tasks like problems, investigations, games and puzzles, that promote mathematical knowledge in a reflective way, and developing creative processes to get solutions.

We only can be creative if we are attracted and challenged by the task. Challenging situations provide an opportunity to think mathematically. Holton et al. (2009) defend the importance of challenge in mathematics classroom when they state: “Students can become unmotivated and bored very easily in “routine” classroom unless they are challenged and yet it is common to hold back our brightest students” (p. 208). A mathematical challenge occurs when the individual is not aware of procedural or algorithmic tools that are critical to solve the problem and seems to have no standard method of solution. So he/she is required to engage in some kind of reflection and
analysis of the situation, possibly putting together diverse factors, therefore having to build or invent mathematical actions to get the solution. Those challenges must respond to the situation with flexibility and imagination (Barbeau & Taylor, 2005; Powell et al., 2009).

Challenging tasks usually require creative thinking. Creativity begins with curiosity and involves students in exploration and experimentation drawing upon their imagination and originality (DFES, 2000). Creativity is typically used to refer to the ability to produce new ideas, approaches or actions and manifest them from thought into reality. The process involves original thinking. According to the Wikipedia creative thinking is a mental process involving creative problem solving and the discovery of new ideas or concepts, or new associations of the existing ideas or concepts, fuelled by the process of either conscious or unconscious insight. For Meissner (2000) there are several descriptions for creative thinking having not a standardized answer. But this author claims that creative thinking may develop as a powerful ability to interact between reflective and spontaneous internal representations. An examination of the research that has attempted to define mathematical creativity found the lack of a consensual definition. According to several researchers (Leikin, 2009; Polya, 1981), we also believe that creativity can be developed if we provide students with tasks that, allowing autonomous approaches, can generate new insights in underlying mathematical ideas. As we said before we are timely in the new national curriculum, so we designed some mathematics tasks fitting 1-6 grades, admitting multiple approaches and providing the development of creative processes of solution. We hope to motivate students to involve in class and challenge them for mathematics activity.

EXAMPLES OF CHALLENGING MATHEMATICAL TASKS FOR THE CLASSROOM

In a constructivist perspective the exploratory tasks in several contexts increase the development of students’ knowledge and mathematical skills. In this setting, while mediator between students and mathematical knowledge, the teacher must offer students diversified tasks that allow them to access mathematical content as well as to highlight and develop mathematical processes such as to experiment, conjecture, investigate, communicate and create, contributing to a more effective learning of mathematics. On the other hand, good tasks must call for mobilization, integration and application of different knowledge. According to NCTM (2000) a task is a good one when it deals or serves as an introduction to fundamental mathematical ideas that constitutes an intellectual challenge to students and allows different approaches.

We present four examples of tasks included in the research plan, of different nature and designed to different grades. Some of them intend to develop number sense while others stress algebraic thinking or geometric and spatial reasoning. However, they have a common objective: to develop new approaches and creative ideas. That is, the tasks must provide multiple solutions in order to raise the student flow of
mathematical ideas, flexibility of thought and originality in the responses. According to students’ age, the teacher may encourage the use of manipulative materials so that children can have an useful involvement in the task.

We briefly discuss possible ways of exploration of the tasks.

**Task 1. Visual Counting - The shells**

*The sea girl organized this way the shells she catched yesterday.*

*Can you find a quick process to count them?*

![Fig 1: Shells](image)

We claim that a previous work with counting tasks in figurative settings can be a particularly good way to develop skills of *seeing* (identification, decomposition, rearrangement) to facilitate similar processes in growing pattern tasks. In fact, in the exploration of growing patterns in figurative sequences it is crucial for students to see the relations among successive terms in order to translate visual patterns into numerical expressions conducting to the generalization process - the heart of algebraic thinking.

Numerical expressions translating students’ thinking in seeing the collection of shells arranged in this manner must be explained. For instance

\[40 = 4 \times 4 + 4 \times 6 = 3 \times (4 \times 4) - 2 \times (2 \times 2) = 2 \times (4 \times 4 + 2) + 4 = \ldots\]

“Four rows of four shells and four rows of six shells” or “Three 4x4 squares minus two overlapping 2x2 squares” or “A 4x4 square plus two above, its reflected image and 4 shells more in the reflection axis”.

The horizontal representation stresses the equivalence of the numerical expressions and allows a new meaning of the equal sign. Our expectations of students’ creativity in this task lay in the different original ways of seeing/counting the number of shells.

**Task 2. Figurative growing patterns - Trucks**

*Observe the pattern.*

![Fig. 1](image)  ![Fig. 2](image)  ![Fig. 3](image)

**Fig 2: Trucks growing pattern**

1. *Sketch the next figure.*
2. What is the area of each figure if the little square is the unit? Write a numerical expression translating a way to calculate that number.

3. Describe with a written explanation how you could construct the figure 25.

4. Describe with words how you could determine the area of any figure of the sequence.

5. Explain, to a fellow that doesn’t believe in your rule, why does it work.

We intend that students look for a pattern, describe it, and produce arguments to validate it using different representations. The previous work with visual counting may help to see a visual arrangement that changes in a predictable form and write numerical expressions translating the way of seeing, in order to make it possible the generalization to distant terms.

Students must be encouraged to observe and see the figures in different ways and to register those various modes in a table looking for a functional relation (Table 1) using more or less formal representation. Creativity can be revealed in the search of different ways of seeing the arrange, to choose the best way to get an expression of far generalization.

<table>
<thead>
<tr>
<th>Number of the figure</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2x3 - 1 + 2 - 3 + 2x2 = \ldots)</td>
</tr>
<tr>
<td>2</td>
<td>(3x4 - 1 + 2 = 4 + 3x3 = \ldots)</td>
</tr>
<tr>
<td>3</td>
<td>(4x5 - 1 + 2 = 5 + 4x4 = \ldots)</td>
</tr>
<tr>
<td>(\ldots)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>25</td>
<td>(26x27 - 1 + 2 = 27 + 26x26 = \ldots)</td>
</tr>
<tr>
<td>(\ldots)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>n</td>
<td>((n + 1)x(n + 2) - 1 + 2 = (n + 2) + (n + 1)x(n + 1))</td>
</tr>
</tbody>
</table>

**Table1: Registration of modes of seeing**

**Task 3. The Euclid’s Game**

*Number of players:* 2

*Material:* - a hundred chart
- overhead projector pens or translucent marks

*Development:*

*Toss who is player 1.* The 1st player chooses a 1-100 number and marks this number on the chart. The 2nd player chooses and marks any other number. In turn, the player subtracts any two marked numbers in order to find a difference that has not yet been selected. The players play alternately until they cannot choose a chart number. The player who can mark the last number wins.

*Play the game several times.* Try to discover a winning strategy.

Throughout games students can develop a greater motivation for mathematics work. The links between the game and mathematics are sometimes surprising and unexpected: the fact that a game has a simple mathematical explanation and its knowledge may entail the possibility of gain provides a good way to enjoy mathematics, its beauty and power. The Euclid’s game is a numerical one in which, with a few persistence, students will be able to discover patterns in the numerical
structure and relations in a flexible manner in order to reach a winning strategy. The fluency of basic knowledge to relate the numeric data collected must give insight to the underlying mathematical concepts in the task and associate them to produce the bright idea of the solution.

The table, which is indicated in the statement with 100 numbers, may be replaced by a table of 6x6, for example, to facilitate and not becoming boring to perform several calculations.

The general conclusion about the game is complex and may be presented as follows: Let n and s the numbers chosen, respectively, by the 1st and the 2nd player. Let $d = \text{gcd}(n, s)$; Let $m = \text{Max} \{n, s\}$. Then the game ends after $m/d$ steps.

It is the evenness of $m/d$ that determines the winner. So if the 2nd player knows how to play, the 1st hasn’t any possibility of winning: if the 1st chooses an even number, it is enough for the 2nd to choose its half to take immediate winning; if the 1st chooses an odd number, it is enough for the 2nd to choose its double, to take immediate winning, or then the successive even number to warrant victory after a number of steps equal to the chosen number.

However, students can take partial conclusions based on particularization with various numbers resulting from several games. For this purpose it is useful to ask questions such as:

- Compare the patterns of numbers marked in each game. Can you explain why some games have so few numbers marked and others so many?
- If the first player chooses the number 26, what number should I choose to be sure of winning?
- If I start the game by choosing an odd number can I win? How?

**Task 4. The Cube Problem**

*From a square sheet of paper, draw the net of a cube with the largest possible volume. Then build the cube by folding.*

This problem involves geometric and spatial reasoning. Many nets can be drawn in a square sheet of paper but there is only one that fits the condition. Fig 3 shows some different attempts students may do to get the solution. This one – the last drawing in Fig 3 – may be achieved either by drawing or by folding. It is necessary insight and divergent thinking to admit one face of the cube divided into four triangles. This requests a novel idea. However, the evolution in the consideration of the different nets as suggested by Fig 3, as well as the intuitive notion of balance and symmetry, may provide this good idea.

![Fig 3: Different attempts to optimize](image-url)
Another promising exploration for elder students is the relation between maximum volume/area. The construction of those successive figures with dynamic geometry software may be a good tool to discover the solution and to verify that it is indeed the optimal one.

**TO CONCLUDE**

Mathematics tasks are not the only feature in promoting mathematical challenge. In this endeavor, the teacher has a critical role in the process of fostering mathematical learning of students. To provide students with challenging situations is in itself a challenge for teachers. They have to choose adequate tasks and help students to carry on with appropriate assistance. This implies a wide and deep knowledge of the mathematics they teach and also a didactical one, in order to interpret what students say and support students who are working on in new situations.

Creativity is new for us as a research field. We expect some results about the impact of teaching and learning elementary students mathematics with the support of these tasks and about if they are real opportunities to obtain creative resolutions that contribute to the learning process. To check these goals we privilege classroom communication including questioning, oral presentations, written work and discussions, as well as focusing on the analysis and comprehension by the teachers of their students’ mathematical thinking.

We intend to present some classroom episodes and partial results in the oral presentation.

**REFERENCES**


Boaler, J. (2002). Open and closed mathematics: Student experiences and understandings. In J. Sowder, & B. Schappelle (Eds.), *Lessons learned from research* (pp. 135-42). Reston, VA: NCTM.


