"YOU UNDERSTAND HIM, YET YOU DON'T UNDERSTAND ME?!!" - ON LEARNING MATHEMATICS AS AN INTERPLAY OF MATHEMATIZING AND IDENTIFYING

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Leaning on the commognitive theory, we attempt to demonstrate how affective, social and cognitive aspects of learning can be treated with the same set of theoretical concepts and analytical tools. This will be done by developing the basic idea of learning mathematics as interplay between activities of mathematizing (talking about mathematics) and subjectifying (talking about the participants of the discourse). We shall exemplify our proposed tools of analysis on a group of usually successful 7th grade mathematics students whose ability to benefit from significant opportunities for learning is shown to be highly sensitive to how they identify themselves and their peers.

Affective aspects of learning mathematics such as attitudes and beliefs have been studied extensively both for discovering differences between groups and for finding relations between those aspects and mathematics achievement (i.e. McLeod, 1992). However, emotional reactions during mathematics learning have received less attention (Leder & Forgasz, 2006), and in particular – their interaction with the cognitive processes of mathematics learning have seldom been researched. Social aspects such as status and positioning in class have received growing attention in recent years (Lerman, 2000), yet less work has been done about the links between social interactions and the emotional and cognitive processes going on within the individual student in class.

Most probably, the lack of studies dealing with the interaction between 'affective' reactions, cognitive processes and social interactions stems from the fact that very specific, dissimilar conceptual frameworks and research tools have been used for studying each of these three arenas. Therefore, the long term goal of our research is to contribute to the effort of creating a unified framework, where cognitive and affective, as well as individual and social aspects of learning would all be seen as members of the same ontological category, to be studied with one integrated system of tools, grounded in a single set of foundational assumptions. In what follows, we report on the conceptual tools we have developed, while trying to address this challenge. We also exemplify the usage of these tools in an interesting episode of social, emotional and cognitive conflict in class.

The proposed conceptual framework leans on the 'commognitive' theory, which recognizes the centrality of communication in all our activities, including the uniquely human forms of learning. This theory, stemming from participationist (Lave & Wenger, 1991) and discursive theories (Harre & Gillette, 1994), maintains that thinking can be viewed as an interpersonal form of communication and mathematics,
being a particular way of thinking, turns to be a special form of discourse (Sfard 2008).

Learning mathematics can be seen as interplay between two concomitant activities: that of *mathematizing* – communicating about mathematical objects; and that of *subjectifying*, that is, communicating about participants of mathematical discourse. Of all subjectifying activities, the most consequential for learning seems to be that of identifying – the activity of talking about properties of persons rather than about what the persons do. Scrutinizing the activity of mathematizing is the commognitive counterpart of cognitive analysis, whereas studying the activity of identifying means attending to all those phenomena that other researchers label with adjectives such as affective, interpersonal or social. The empirical material presented in this article demonstrates how the activity of identifying may interfere with the activity of mathematizing, and thus with the learning of mathematics.

**THE STUDY**

The study was held over a period of 5 months, in an extra-curricular program, where one of us taught three groups of 7th grade students: students with very high scores in mathematics, with moderate-to-high scores, and with low scores. In each group there were 4 students, 12 students overall (7 boys and 5 girls). The present event occurred during the 11th lesson with the moderate-to-high achievers group. This group included two boys - Ziv and Dan, and two girls - Edna and Idit. Generally speaking, Ziv and Dan had a history of high achievement in mathematics, Idit was generally successful though in some areas she encountered problems, and Edna had usually moderate scores.

The lessons were video filmed by 3 stationary cameras, directed at the student's front. Additionally, all written material performed during the lesson was collected. The recordings were transcribed in Hebrew, and the examples given here were translated into English by the authors.

The present episode was chosen for deep and thorough analysis because of the very high occurrence of subjectifying utterances that accompanied what appeared to us as an inexplicable block in the advancement of the group towards a solution of a given mathematical problem. The students were presented with a worksheet containing a problem called the Chocolate Factory Problem (Figure 1). Its purpose was to help students to advance toward a discourse on fractions (though fractions were not explicitly mentioned).

What mystified us was the fact that the participants seemed impervious to one student's (Ziv) seemingly lucid and cogent explanations and that the eventual eye-opening effect came from what looked to us as a much less coherent and quite opaque argumentation of another student (Dan). In what follows we show that much more than scrutinizing the flow of the "mathematical content" of the conversation is necessary in order to understand what blocked the learning process.
The Chocolate Factory

The chocolate factory produces four different types of chocolate bars, all of the same size, as shown in the drawing.

When bars get damaged, the factory can’t sell the chocolate in its original package. The factory donates the damaged bars after repacking them in bags. Each bag contains the amount of chocolate equal to one type-D piece and two type-B pieces.

1. In how many different ways can one pack such a bag?
2. What are these ways?
3. How can you be sure you have found all the possible ways?

Fig. 1: The Chocolate Factory problem

Mathematizing

While participating in mathematical discourse, interlocutors combine mathematical keywords and mediators into mathematical objects. Indeed, to act as a competent participant in mathematical discourse, one has to realize (translate) words such as numbers, functions or sets with the help of other mathematical words and mediators. A mathematical object is a mathematical signifier together with its realization tree, a hierarchically organized set of all the realizations of the given signifier, together with the realizations of these realizations and so forth.

One way to analyze the activity of mathematizing is to follow the flow of mathematical objects, that is, to try to identify the ways in which the participants realize the focal signifiers at different points in the process of solving the problem. One important question that can be answered on the basis of this information is that of the effectiveness of communication: We can decide in a systematic, testable manner whether two interlocutors are speaking of the same object while using the same words.

Let's take a close look at how two of the students, Ziv and Dan, realized the signifier a bag[containing] the amount of chocolate equal to one type D piece and two type B pieces (which we shall shortly term 'the required bag', or simply 'bag') while trying to solve the Chocolate Factory problem.
Dan's first try

As evidence by numerous utterances during the first minutes of the episode, Dan and the two girls had considerable difficulty trying to understand the question. Apparently, they were unable to unpack the complex 'bag' signifier so as to be able to start thinking about any realization procedure. After grappling with the question for three minutes, a breakthrough occurred and Dan offered an idea:

Dan: so it’s actually they say that they mean this.. that one like this, each piece of one type, of type D, equals two of type B [pieces] right? (lines 233-235)

Dan had arrived at the (wrong) impression that the word *equal* in the description of the task signify the relation between the amount of chocolate in a type-D piece and the amount of chocolate in two type-B pieces (in other words, $1/3 = 2/5$). Not surprisingly, this interpretation stymied his further attempts at solving the problem. Indeed, from here, it was not clear what the words "number of different ways to pack such bag" could possibly refer to. His idea was questioned by the teacher, who encouraged Ziv to offer his alternative solution.

Ziv's try

Ziv remained silent throughout the conversation between Dan, the girls and the teacher. During the first 10 minutes of the episode he worked on his own solution. When he eventually spoke up, his proposals flowed one after another, contributing to the gradual emergence of a coherent realization tree for the requested 'bag'.

Ziv: O.K. (..) the first way (.) is the way they showed us, which is one stripe from here (points to Bar D) and two from here (Bar B, see fig 2. No. 1), that’s the first way¹. The second way is to take all these (Bar C) and two from here (Bar B, see No. 2). A third way is to take a=ll² these (Bar A), two from here (Bar B, see No. 3). Now let’s move here (points to Bar C). A third way is to take a=ll these (Bar A), and all this part (Bar C, see No. 4), all the six-, all the-, these

Fig 2. Ziv's realizations of the requested 'bag'.

In spite of the ostensive clarity of Ziv's explanations (supported by suggestive pointing), his classmates claimed they were unable to understand what he was saying. This left space for Dan's new take on the problem.

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¹ The visual realizations of the students’ words are our interpretations, based on the way the students pointed on the worksheet
² The transcription signs are as follows: = marks - a prolongation of the syllable; (.) marks a very short halt.
Dan's second try

After his 2B=D proposal was rejected by the teacher, Dan went back to coping solitarily with the problem, hardly paying attention to Ziv's solution. He then volunteered to explain his own solution to the girls and set out into a long and somewhat muddled explanation.

Dan: Uh- the chocolate factory did, um- things-, ah like, did a defected [pack], and did that the defected chocolates pack, will be this size, O.K? (Pointing up and down a D rectangle, see fig. 3-1) .. and in this size (fig 3-2). Right? Wait. (Turning the worksheet towards himself) If it’s actually in this size, then why? Ah, I got it. ‘nyway he did in this size, O.K? (pointing to D, fig 3-1) And in this size (pointing to B, fig 3-2). Two different packs. Now, they have a few types of packs, and they wanted actually, each, each pack of the defected [packs] they wanted actually to make it into such a pack (pointing to D, fig 3-1). They only have two types of packs, that they can get, and they can’t change the size of the packs, so they have to insert [it]. Now, here (pointing to bar A, fig 3-3) there are many small ones, so they can actually divide them so they be put into such [a pack] or such [a pack] (fig 3-1 and 3-2). That’s why they asked you here, if you can pack such a bag with squares of type A alone, so yes… So actually you can divide them so they'll be in this (pointing to bar D, fig 3-1), or like this (bar B, fig 3-2) because they are bi-, they're small. (Lines 536-554).

Fig 3 – The diagrams Dan was pointing to (the pointed parts are shaded gray)

Apparently Dan had arrived at a realization of a 'bag' he was quite content with (though it was wrong, again). He figured there are two 'legitimate' types of chocolate bars: one of the size of a single D piece, and one of the size (shape) of two B pieces. Thus, his realization tree for the requested 'bag' was as depicted in fig 4.

Fig 4 – Dan's realization tree for the signifier 'bag containing the amount of chocolate equal to 1 type D piece and 2 type B pieces'

Surprisingly enough, the girls, who earlier insisted they 'don't understand a thing', now showed signs of comprehension. Idit claimed she understood Dan, and Edna offered: "and then it's like ten cubes" (line 557).

What was happening here? These students seemed to be learning from the participant who himself was struggling for understanding, and who offered a realization which
was much too blurred and ambiguous to be helpful. All this happened after they let the obvious opportunity for learning slip away – after remaining unimpressed by a solution which, according to the observers, was not only correct, but also presented quite clearly and convincingly. Although nobody seemed to doubt the correctness of Ziv's solution, no visible effort was made to find out what his proposal was all about. We found no evidence for the other students being even interested in Ziv's explanation.

In order to understand what was happening, it was necessary to look into what we claim is a process that always runs in parallel to 'mathematizing', the process of 'subjectifying' and, in particular, 'identifying'.

SUBJECTIFYING

Subjectifying could be about oneself (1st person subjectifying), about others (3rd person subjectifying) or stated explicitly toward the subject of the communication (2nd person subjectifying). Following Sfard & Prusak (2005) we shall denote every subjectifying utterance with a specific notation that will clarify who the informer, the subject and the receiver of the communication are. Thus the notation EdnaZivTeacher will signify an utterance about Ziv, made by Edna and directed at the Teacher.

Subjectifying acts can be verbal or non-verbal (such as in gestures, facial expressions or meaningful intonation of the voice). They can also be direct or indirect. Direct subjectifying utterance refers directly to the subject (such as Dan (to Ziv): "no one can understand you"). Indirect subjectifying are acts that can be interpreted as being about one of the participants, even if the object of the act is not stated explicitly (such as when Ziv says about the other student's failure to comprehend "so they should try harder", indirectly stating he himself is superior to the others in his mathematical performance: in other words, this utterance may be marked ZivZivB).

Moving from subjectifying to identifying

By using the term 'identity', we refer to Sfard & Prusak's definition of identity as a narrative. More specifically,

"identities may be defined as collections of stories about persons or, more specifically, as those narratives about individuals that are reifying, endorsable, and significant" (Sfard & Prusak, 2005, p. 16)

Sfard & Prusak made the distinction between narratives of current identity which consist of stories about the actual state of affairs, and designated identity "consisting of narratives presenting a state of affairs which, for one reason or another, is expected to be the case, if not now then in the future." (p. 18). This distinction will serve us when we look at different designated identities the participants are attributing to one another during this episode, and how such designated identities might conflict with the current 1st P identities the participants hold.

One of the most problematic obstacles for studying the mechanisms of identity building is the inaccessibility of some of the ‘identity narratives’ during a non-
intrusive observation of students in class. Our first task, therefore, was to prepare an operational means for extracting ‘identifying processes’ (processes of identity building) from the natural discourse of the students in class and, in particular, for deciding which subjectifying utterances could count as identifying. We did this by assessing how general the subjectifying message is. At the lowest level of generalization we placed utterances in which an interlocutor evaluates what she just did or is about to do (e.g., Idit: "I didn't understand a thing"). A remark about a general characteristic of the speaker's or other person's participation (such as Edna: "When Ziv speaks I never understand him") was classified as representing a higher level of generalization. We decided that the highest level of reification occurs when rather than assessing the participation (what people do), the speaker evaluates properties of a person (what the person is or has) or of this person's memberships (with whom the person belongs), e.g., Dan (to Ziv): "You'll never be a teacher". Such reifying utterances are identifying by definition. However, general participation evaluation and even specific participation evaluation utterances can sometimes add up to form a coherent identifying narrative, provided they are recurring and consistent.

Most researchers interested in students’ identities rely in their analysis exclusively on students’ verbal 1st person identification, such as those obtained in interviews. And yet students of this age seldom talk extensively about themselves, and thus these are the non-verbal and indirect subjectifying actions which may often be a major, sometimes the only, ‘window’ to students’ first-person identities. In our analysis, we found that when examined in the context of verbal direct identity narratives, repetitive, consistent non-verbal and verbal indirect subjectifying actions may often provide the most valuable information about how the students view themselves and others.

Additionally, emotional declarations and gestures are key signs of significance, and thus indicate which narratives and activities are important for the interlocutors. It can be reasonably assumed that whatever an emotionally loaded statement is, it has something to do with the student's identity – with how the student sees herself in the longer run. For instance, when Ziv solemnly hides his face behind his worksheet, avoiding eye contact and sitting ubiquitously quiet after remarking that Edna has 'understood' Dan and not him, we may assume he's emotionally hurt and that 'being understood', and 'explaining better than Dan' is important for his 1st P identity.

To summarize, we will define identifying utterances (which may be either verbal or non-verbal) as those subjectifying acts that signal that the identifier considers a given feature of the identified person as permanent and significant.

Technique of analyzing identifying actions

In order to understand the identifying processes going on in this episode, we extracted all the subjectifying verbal and non-verbal acts made by the participants. These utterances were inserted into four tables (one for each student), each table
containing all the group member's references toward the subject of the table. Looking at the tables, it appeared immediately that most of the subjectifying actions were made about Ziv. His table was the richest in subjectifying utterances in general, and in 2nd and 3rd P utterances in particular. In other words, not only was Ziv the participant who talked directly about himself the most, he was also the one who got the most attention from the other participants. As we shall shortly see, this attention was far from being positive, at least on the part of the other students.

In the following table\(^3\), we bring examples of the most telling subjectifying acts from Ziv's table, those which can be classified as 'identifying' according to the definition made above.

**Table 1 – Ziv's subjectifying acts**

<table>
<thead>
<tr>
<th>Speaker, object and addressee</th>
<th>What is said (and how)</th>
<th>Subjectifying category</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teacher; ZIV; Ziv</td>
<td>Teacher: [explain your solution to the others] because you're the only one who understood [the question] (line 99)</td>
<td>Verbal 2nd P specific participation evaluation</td>
</tr>
<tr>
<td>Dan; ZIV; Ziv</td>
<td>Dan (quietly, while Ziv starts explaining again): Enough, Ziv, you won't be a teacher. (line 678)</td>
<td>Verbal 2nd P designated identifying</td>
</tr>
<tr>
<td>Edna; ZIV; Ziv; Teacher</td>
<td>Edna (to the Teacher; in an annoyed voice): He just- he talks to me like I'm his [little] girl! (to Ziv, mummifying angrily a 'teaching voice') Ziv, you understand that if it's ^so and so^ than it's this and that? Yeah?! (Turning back to the teacher) That's how he talks to me! (line 704)</td>
<td>Verbal and Non-verbal identifying of a complex type: Edna speaking of how she thinks Ziv is identifying her.</td>
</tr>
<tr>
<td>Idit; ZIV; Teacher; Others</td>
<td>Idit: No [I didn't understand Ziv]. Ziv talks to himself and he thinks everyone is listening to him so he (mummifying a 'babbling' voice) &gt;pa pa pa pa, wa wa wa wa wa&lt; (line 443)</td>
<td>Verbal 3rd P General participation evaluation and non-verbal identifying</td>
</tr>
<tr>
<td>Ziv; ZIV; DAN; Edna</td>
<td>Ziv (to Edna, in an annoyed voice) Ah! Like you understand him [Dan], and me you don't? (sits back and looks hurt: covers part of his face with the worksheet, avoids eye contact and seems to 'close up') (line 555)</td>
<td>Verbal + Emotional-Non-Verbal 2nd P specific participation evaluation, highlighting the rivalry between Ziv and Dan.</td>
</tr>
</tbody>
</table>

Looking at this table, and from the analysis of all the subjectifying acts in the transcript, a clear picture of the identifying processes going on in this episode emerged. Let's first look at Ziv’s 1st P identifying. It included a few main themes:

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\(^3\) Out of space restrictions, the structure of this table is different from the one used for our analysis. The analysis table used a column for each participant, where all his/her utterances about Ziv were documented.
First, Ziv made it clear that he thought himself superior to the others (especially to Idit and Edna). Second, he was very competitive with Dan (for instance by trying to ‘beat’ Dan and insulting him when he attempted to catch up). Finally, he seemed to get very hurt when the girls failed (or refused) to understand him conveying that 'being a leader of the mathematical discussion' was an important part of his identity.

The Teacher’s 3rd P subjectifying of Ziv was very consistent, verbal and direct, and thus it could safely be claimed she was identifying Ziv as the most competent participant (mathematically speaking) in the group, and as one who could, and should, help the others understand the task.

The girls (both Edna and Idit) consistently identified Ziv as an "Incapable teacher" who may be smart but does not know how to explain. He was identified directly by Edna as condescending and admittedly complied with this identity, for instance by rolling his eyes at Edna's disability to comprehend him (though all the students had long histories together, thus it might be that the interaction seen here was rooted in the long past of their acquaintance).

It can be reasonable to conclude that the combination of Ziv’s arrogant behaviour and ‘superior’ 1st P identity as enacted by him during the lesson - combined with the “he’s the competent leader” 3rd P identity afforded to him by the Teacher - created a clear resistance from the girls to learn anything from him. Look at Edna, for instance, at the point when Ziv was trying to explain his solution to her (the 3rd time around) and she aggressively rejected it (see table 1, line 3). From her point of view, at this particular moment, she was offered a very undesirable identity by Ziv, and indirectly by the teacher - that of the 'little, inferior', perhaps even 'incapable' student. If she had cooperated by 'understanding' Ziv, she would have simply approved this identity, something she probably very much wanted to avoid.

Dan, on the other hand, being confused himself and showing no condescending behaviour, was much less threatening. In other words, his acts offered no endangerment for the girls designated identities as 'intelligent learners'.

To conclude, what we see here is a struggle of identities. As a result of this struggle, something very basic was missing from the students' discourse which would enable a useful learning process: a teaching-learning agreement. Sfard (2008, p. 283) coined this term for the implicit understanding, formed between two or more interlocutors, that one of them is the 'teacher' who has the authority to determine what is 'true', and the others are the 'students' who 'learn'. In our case, it is clear that the teacher believed such an agreement should exist, and as Ziv was making signs he had solved the problem, it was natural from her point of view, that he would take the role of the 'teacher'. However, as this agreement would endanger the designated identities of the girls, they strongly rejected it, even at the price of blocking their advancement in solving the presented problem.
CONCLUSIONS

This case shows how powerful processes of identifying may be hindering mathematical discourse. The way the student chooses to participate in the mathematical discourse is affected not only, and perhaps even not mainly, by her mathematical competence, but also by her 1st P identity. Of principal importance here are her designated identities constructed by herself and by other participants.

It should also be noted that it is not always the 'weak' or peripheral students who pay the social price during participation in mathematical discourse. As Ziv's behaviour shows, it is the strong student who would sometimes be hurt.

Finally, teachers' actions, even if well-intended and performed in the attempt to advance the mathematical discourse in the class, can fuel and shape counterproductive identifying interactions among the students. As we have shown, such identifying interactions may hinder the learning process, achieving an effect exactly the opposite to the one intended by the teacher.

REFERENCES


